# THE MATHEMATICAL GAZETTE

EDITED BY
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T. A. A. BROADBENT, M.A.

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VOL. XXI. NOVEMBER, 1937. No. 246

## BRITISH ASSOCIATION NUMBER.

THIS number of the *Gazette* is devoted to reports of the proceedings of Section A* of the British Association at Nottingham, September 1937.

On Saturday, 4th September, a discussion on "The Unification of Algebra in the Schools" took place, under the chairmanship of Professor E. H. Neville; the speakers were T. A. A. Broadbent, G. L. Parsons, M. H. A. Newman and W. J. Langford.

On Tuesday, 7th September, a discussion on "The Bearing of Higher Geometry on the School Course" took place, under the chairmanship of Professor H. T. H. Piaggio; the speakers were E. H. Neville, H. G. Green, W. H. McCrea and H. S. Ruse. The discussion was followed by a paper on "Hypersolid Concepts, and the Completeness of Objects and Phenomena", with photographs and models, by Cyril H. H. Franklin.

## BRITISH ASSOCIATION, 1938.

THE 1938 meeting will be held at Cambridge; attention is drawn to the earlier date of the meetings, 17th-24th August.

# THE UNIFICATION OF ALGEBRA IN SCHOOLS.

*4th September*, 1937.

Chairman, Professor E. H. NEVILLE, University of Reading

**Professor T. A. A. Broadbent** (Royal Naval College, Greenwich):

AN INTRODUCTORY SURVEY OF THE PRESENT SITUATION

My function in this discussion is the humble one of attempting to supply a rough scaffolding into which the durable parts of the edifice will be fitted by the other three speakers.

The teaching of algebra has never been dominated by a single work in the way in which Euclid for so long a time dominated the teaching of geometry. For that reason, perhaps, the revolts in geometry have corresponded to quiet reforms in algebra, and the problems involved, though equally important, have never received the same amount of attention. On the teaching of the subject in schools, excellent books exist; there is Sir Percy Nunn's classic, and more recently, a book by Durell, and much relevant matter in Godfrey and Siddons, *Teaching of Elementary Mathematics*. Even so, it is not easy to obtain a comprehensive view of the subject in its development through the school to the university, nor has the impact of recent trends in research work on the demands of the universities been fully investigated. As part of this morning's discussion will be concerned with this impact, it might perhaps be worth while clearing up one point. The idea that the university teacher spends the whole of his time in training professional mathematicians is an illusion; in the modern universities he spends a good part of his time training those whose interest in mathematics is subsidiary to their interest in other subjects or to their interest in the process of obtaining a degree. When we speak of what the universities expect from the schools, we mean simply what the universities rightly expect from those whose last years at school have been mainly given to mathematics. Most university teachers understand that this question, though naturally the one which most interests them, is by no means the whole or even the larger part of the problem of the teaching of mathematics in schools.

The teaching of serious mathematics in the universities is influenced very considerably by the kind of equipment demanded by research work in mathematics or in subjects which rely on mathematical methods. This influence in turn bears on the training of scholarship candidates, so that eventually the sixth form feels, indirectly, the weight of the Royal Society and the London Mathematical Society. The time lag, however, is very considerable; it may be part of to-day's discussion to determine whether that lag should be decreased, whether it can be decreased, and how it can be decreased, for algebra is particularly concerned in this process at the moment. Let me take two examples. Recent work in mathematical physics, for instance that of Eddington, Dirac and Milne,

has shown an increased dependence on algebra ; not so much on the dexterous manipulation of complicated expressions as on the apparently simpler but much more sophisticated and subtle axiomatic algebra, such as is developed in, say, van der Waerden's book rather than in the older classical treatises of Chrystal, Serret or Weber. Physics, in fact, no longer deals with numbers, but with alphabet numbers, $q$-numbers, E-numbers, algebras indeed rather than algebra. Again, a quarter of a century ago, the proper equipment for pursuing original work in algebraic geometry was mainly a thorough acquaintance with algebraic functions and their integrals ; now, although this work is as important in itself as it was then, the importance of axiomatic algebra in this domain has increased enormously. Changes of this type have had their influence on Tripos papers, which, as far as algebra is concerned, look very different from those of ten or fifteen years ago. Mr. Newman will be able to give us some idea of how far he would like this influence to appear at the scholarship stage, and Mr. Langford will tell us what he thinks can be done in this direction. The primary problem is one of time. If we want to put more in to the school programme, it must be at the sacrifice of something already there.

Now modern practice has already shown how a good deal of time may be saved at this stage by the breaking of the artificial barriers which used to be set up between various mathematical domains. Chrystal denounced the artificiality but the first effective step towards removal was probably the account of the logarithm as defined by the integral in Hardy's *Pure Mathematics* and the adoption of this method in the influential and popular school texts by Durell. Now that the calculus plays such a prominent part in the Higher School Certificate course, there is no need for tedious and long-winded dodges to avoid using it ; the wastage is less apparent in algebra than in coordinate geometry or dynamics, but it may be that all the possible economies have not yet been completely exploited. Are there any advantages in teaching partial fractions before the stage at which integration makes their introduction inevitable and natural? How many of the manipulations of the binomial theorem are tedious as algebra, but easy and pleasant as examples on the calculus?

Mathematics is an economy of thought ; let us beware of over-ornamentation. I have helped to solve problems of dinner-plans and of selecting cricket elevens, but never by the methods which occupied so much of my time as a schoolboy, those stupid and elaborate questions which must have blinded many a pupil to the real importance of permutations and combinations. Facility of manipulation is necessary but if acquired at the expense of a sound grasp of fundamentals it becomes a weakness. Too often do the older school-books on algebra give the impression of a collection of watertight compartments, beautifully constructed, but put together without much regard for the final structure of which they are the component parts. Moreover, watertight compartments are apt to be dry ; so is manipulation without understanding.

Naturally, the other extreme is equally to be avoided, since an appreciation of ideas without the technique enabling them to be used simply leads to sterile frustration. Our ideal must be the golden mean, to be attained by economy of thought and simplicity of ornament.

It should be said that what we are doing here this morning is not proposing a revolution, but simply describing and attempting to coordinate changes that are actually taking place. For evidence, let me quote from the preface to a recent book on algebra written for pupils in their last terms at school: the authors "believe that many of the topics which constitute the conventional course are only of value in so far as they illustrate general ideas, and that much of what has been called 'higher algebra' in the school course should be scrapped". These are the words of two of our best-known teachers of school mathematics; and they add, "Only the requirements of certain examinations have prevented them from pursuing a more drastic policy than they have actually adopted". This, I take it, is not more than an attempt to break the vicious circle in which we teach what is asked in examinations and ask in examinations what we know is taught. Eventually this process tends to classify boys into two compartments—the good pupils, who will do what they are told in spite of not understanding it, and the bad pupils, who will not do what they are told because they do not understand it.

There was at one time a very marked tendency towards such a classification in the early stages of teaching algebra; in order to foster manipulative ability, great stress was laid on algebraic addition, multiplication, factorisation, simplification, and—may one add—distraction and uglification, while matters which present real difficulty to the average pupil, the passage from arithmetic to algebra, the extension of the number field, the index laws were treated with a rapid formalism and too great a reliance on the maxim "Go on, and faith will come to you". Mr. Parsons is going to talk about this stage of algebra teaching, and will make it plain that the approach to algebra by the formula or by the equation, the later introduction of a reasoned account of extension of number fields, and so on, contribute not only, though most importantly, to a right grasp of ideas rather than of processes, but ultimately to a better appreciation of process and to a more kindly attitude to the axiomatic algebra of the last year at school if the pupil is one who reaches that stage.

Finally, I have left to the end what seems to me to be the most difficult question of all. What should we do about the vast majority of schoolboys who will view the School Certificate with some relief as constituting their last encounter with algebra? The ideal is obvious enough; to impart a knowledge of ideas in connection, of method rather than of methods.

But we need not be too ambitious; is it certain, for example, that the average schoolboy's knowledge of the history of his own country

at the School Certificate stage is sufficiently profound to enable him ultimately to cast an intelligent Parliamentary vote? Why, therefore, should we expect more from a subject that is more difficult and more repugnant to the ordinary pupil? To think clearly is one aim; to obtain the School Certificate is frequently another and a different aim. Are we quite clear ourselves as to what our aim for these pupils is? Perhaps we should all be satisfied if at that stage some appreciation of algebra as a science had been acquired. In the larger schools, the flexibility obtained by the system of sets makes it likely that good results in this direction can be and are being reached. But what is to be done in the small schools, where the set system is impossible, where problems of organisation and of curriculum are made much more thorny because the staff is small? I hope that Mr. Langford and Mr. Parsons will be able to suggest some solution of this problem but it seems to me to be so difficult that if I am disappointed I shall not be surprised.

**Mr. G. L. Parsons** (Merchant Taylors'):

### The Introduction of the Fundamental Ideas of Algebra

I. My part in this symposium is to deal with the introduction of the fundamental ideas of algebra, as far as the stage usually known as the "Certificate stage". As I shall attempt to show later, this is an extremely vague phrase, but it will be sufficient for the present to note that I am speaking of that part of algebra which is a compulsory subject for the great majority of secondary school pupils and that it is no part of my plan to deal with the needs of universities or any specialised developments of the subject. These needs, in my opinion, should not be allowed to influence unduly teaching at the "compulsory" stage.

It will not have escaped your notice that a very prominent educationalist has on several occasions suggested that Algebra is one of the subjects that might well be dropped out of the curriculum. Be this as it may, and it is a controversy into which I do not propose to enter to-day, it expresses a wide-spread dissatisfaction with the teaching of the subject as it was. The revulsion from the previous presentations of the subject is not yet however complete and the present situation is that several methods of presenting the initial subject matter still contest the field. At the risk of preaching to the converted, I propose to consider these in some detail.

These are three in number, *viz.* (i) the traditional "Four Rule" method, (ii) the "equation" method and (iii) the "formula" method. I shall further consider the introduction of "directed" numbers, graphs and functionality.

II. The Four Rule method, which is no doubt familiar to all of you, commenced with the evaluation of complicated expressions, followed by the immediate introduction of negative numbers. The methods of

Addition, Subtraction, Multiplication and Division were then studied (including of course Long Division), together with all the rules for manipulation of brackets, etc. Set rules for the various operations required were given in the text, usually without any reference to the reasons underlying them and it was not till all this had been done (often with the addition of Fractions) that the Equation and the problem were reached. [One sentence from the explanation of an equation is perhaps worth repeating . . . " the *letter* whose *value* it is required to find in any equation is called the unknown *quantity* ".] Indeed it is not until after many wearisome pages of simple equations of every possible (but unlikely) type that we reach the statement (I quote from a popular text book published between 1880 and 1890) that " by a single algebraical formula we are enabled briefly to express a whole class of results . . ." etc., an idea which occupies a much earlier place in present day practice.

A glance at the list of the contents of this textbook will serve to make the point clear. An analysis of the contents of the same book by pages is also interesting. Of some 208 pages (bookwork and examples) devoted to what might justly be called " Elementary " work, some 125 are devoted to pure manipulation, 49 to the solution of various types of equation (up to quadratics) and 34 only to symbolical expression and formulae. Nor was this balance redressed at a later stage for a similar study of a " Second Course of Algebra " of about the same date, shows 144 pages of manipulation, 48 of equations (including the theory of quadratics which is largely manipulative work) and 32 pages of applications in the form of problems.

This study of technique for its own sake, this formal emphasis on manipulation, is now happily recognised as a sterile and barren thing. The pursuit of a manipulative skill of a degree far beyond that for which he is likely to find a need is not the sort of object which is likely to appeal either to the pupil or to the teacher. It is like giving a man a training in all the classical forms of architecture and then setting him to design a chicken-run. The study of the subject in this manner benefited nobody (except perhaps the journalist, who, when in need of a symbol for the height of incomprehensibility, has only to think of what comes at the end of the textbook of the most unintelligible subject he studied at school—and so he writes " as incomprehensible as the Binomial Theorem "). Successful as such a method may have been in passing examinations, the proof that it had no permanent value is not far to seek, for we must all be familiar with some one who tells us " I was good at algebra at school, but now I don't even remember what it was about ". [The fact that we ourselves survived and profited by such instruction is, of course, to be credited rather to our own pig-headedness than to any excellency inherent in the method.]

III. The other two methods of introducing the early stages of the subject illustrate a change in teaching practice analogous to that which has taken place in the teaching of geometry. The emphasis in both these subjects has been shifted from logical structure to

subject matter. The logical structure of algebra, viz. the use, development and manipulation of notation, must come in, but comes in only so far as it is required in working out the subject matter (problems, equations, formulae) and each new departure is introduced—not by a series of arbitrary rules—such as those italicised sentences with a heading in black type, which epitomised the various processes in the algebra books of a quarter of a century ago—but by examples which show up the convenience and desirability of such extensions of notation.

It is customary for the earliest algebraical notations to be embodied in the arithmetic course ; these should be confined to the easiest use of letters for numbers and the index notation. The simplest ideas of manipulation may be used—thus the formula $A = lb$ for the area of a rectangle may be transformed into $l = \frac{A}{b}$, $b = \frac{A}{l}$. Similar ideas may perhaps be employed in the study of fractions where such forms as $2 \times \frac{m}{n}$, $\frac{3m}{3n}$, $\frac{2}{3} \div a$ and so on might be considered.

IV. When the study of algebra proper is reached, there are two schools of thought as to the best procedure. In the view of one of these, algebra proper begins with the study of equations. These are considered first as " hidden number " problems and very soon as the numerical expression of the conditions of some problem of a practical kind. After the preliminary work in solution of equations, a variation of the conditions of the puzzle or problem will make clear the necessity for various manipulative devices [*e.g.* brackets—indices—trinomial products and so on] and some time is spent " by the wayside ", so to speak, in the investigation of the laws regulating these forms of notation, a return being made to the solution of problems of a type similar to that from which the necessity for the new notation arose. Fractional equations are introduced quite early and this leads to a discussion of manipulation of fractions in general. Thereafter the subject follows the usual course. There are two necessities on which teachers following this method (or indeed any method) would be wise to insist. First, when a new topic is introduced, it should be introduced by a concrete problem. [Thus where our old textbook had first, 4 exercises embodying all the types of simultaneous equation and then an exercise on problems, the practice suggested is first the enunciation and solution of a suitable problem, followed by further practice in the solution of similar equations. Later other problems introduce more difficult types of equation, including easy simultaneous types.]

The second point on which emphasis should be laid is the need for clear and concise expression. At all times, during this stage of the work, letters must represent *numbers* and not *quantities* or *numbers-of-things* : answers to equations must be carefully checked and the check written out in a logical way, proofs that $0 = 0$ being carefully avoided and strictness in the use of the = sign being insisted upon.

Phrases like " taking over the term and changing its sign ", " cross-multiply " and the like may be allowed only when the pupils understand that these terms are generalisations of their own experience and can explain their meaning if called upon to do so. In this connection, I can also recommend a rule enforced by one of my colleagues who insists that answers to problems shall appear in the exact phraseology of the question, which makes his algebra periods a valuable part of the English teaching of the school. Care should be taken not to introduce negative numbers inadvertently, *e.g.* equations like $5-x=x-11$ should be avoided at this stage—for although the solution $x=8$ is easy enough to get, its verification involves the two forms $5-8$ and $8-11$ which can hardly be said to be identical until they possess a meaning. It is, of course, quite easy to lead into a discussion of directed numbers by a deliberately organised dilemma of this kind, but there are other and better ways.

V. Other teachers believe in a slightly different approach. They commence with an extension of the previous experience of the pupils in the use of numbers for letters and develop the notational aspect of this by familiar examples in everyday use or from other branches of mathematics. Good pupils readily take to the search for effective and economical symbolism (*e.g.* " $e$ is a number lying between 2 and 3 ", etc.). The next stage is to lead them to make up their own formulae as generalisations of arithmetical results—such statements as " 3 times 7 + 5 times 7 = 8 times 7 ", " $2\times7+1$ is an odd number " being discussed and generalised. The use of brackets may also be brought in by examples of suitable type. Questions on perimeters and angle-sums may easily be adapted so as to give use to and illustrate the rules for addition of like terms, while areas may be used in the illustration of products, indices, and in the easy cases of the evaluation of expressions, such as $(2x+3)(x+2)$ (illustrated by a diagram). Examples may be made up to illustrate such identities as $a-(b-c)\equiv a-b+c$. [A man has £$a$ in the bank : he draws out £$b$ and puts £$c$ in again : how could he do this more easily ?]. And when these rules have been well explored and explained they may be regularised for future reference. Equations and problems are then reached and the two methods then merge into one.

The divergence between these two alternative methods of procedure is not so great as appears at first sight. The formulation of equations to satisfy the conditions of given problems is merely a variation of the manufacture of formulae, and, on the other side, the consideration of formulae leads necessarily to the consideration of conditions under which the formulae would yield certain results—and that is the solution of equations. Many teachers use, in practice, a compromise between the two methods, a procedure which is suggested in the Report on the Teaching of Algebra, issued by the Mathematical Association. It is possible that the " equation " method is easier and therefore more suitable for " dull " pupils. If it is adopted, the illustrations and problems should be chosen from

as wide a field as possible, to obviate the idea arising that a letter merely represents a fixed but undiscovered number in an equation—an idea which makes it difficult to develop the notion of functionality later.

VI. A prominent feature of both the modern methods of algebra teaching is afforded by the question of " directed " numbers. The study of these is now left till later in the course—possibly till after the first encounter with Simultaneous Equations and Graphs. Previously they appeared at a very early stage in the proceedings, accompanied by rules for their use—these rules having, presumably, been provided by a thoughtful providence along with the Decalogue and Kennedy's Latin Primer. No question of justifying these rules need exist—and the existence of the numbers themselves was amply proved by their appearance in the textbook. Now I am a mentally lazy individual, prone to believe what I am told (unless it involves manifest inconsistency) and I can not remember having any serious qualms on this matter. But there is no denying that the average pupil found, and still finds, difficulties in it. There is a primary difficulty to be overcome before we become involved in difficulties arising out of the manipulative technique required for such numbers, though it is true that a good deal of explanation and drill will be necessary to get that technique established. But the first difficulty to overcome is that of the two-fold function of " zero ". Zero-ness in signless algebra (such as we have been discussing up to the present) is a concept taken from arithmetic, but we now require to use " zero " in quite a different way as a standard mark from which measurements are taken.

Some preliminary discussion of this use of 0 is of great value in the early stages of the discussion of directed numbers. Such apparatus as the thermometer will be useful in this discussion—few children will imagine that when 0° C is reached, zero-ness of temperature in the signless sense is achieved. This can lead on to other illustrations of the use of an arbitrary zero, such as sea-level, variation of the barometer, boys' marks in relation to the average, " standards " in the school sports and so on. Not till this sense of zero to denote normality (so to speak) is quite clear should any explanations of manipulations with directed numbers be attempted. And no fear of wasted time need be entertained, for in doing this we shall be repaid in many other parts of the work, especially in graphs, where pupils often spoil their work by an unnecesary attempt to bring absolute zero into the picture.

The devices used for illustrating and establishing conviction of the rules for the use of directed numbers will naturally vary with the age and intelligence of the class. With a very young class, it is quite a good method to mark a scale of numbers on the floor and to make a boy walk along it, while the rest of the class records what is happening. Whatever external method is adopted, it will be found advisable not to use the signs + and − as directive symbols in front of numbers in the early stages. I have found it useful to use red chalk

for negative numbers and white for positive and the class had no objection; in fact, I had not nearly so many sorts of numbers as they had coloured pencils! The rules for the addition and subtraction can then be dealt with, as follows: " + and − are signs of operation which tell you which way to face: + means look up the scale of numbers; − means turn round and look down it. White (*i.e.* + ) numbers mean steps *forward* and red (*i.e.* − ) numbers means steps *backward* (which ever way you are looking). If no sign of direction is given, assume +.

A few exercises whose answers are already evident are taken first to establish faith in the scheme and then by further exercises it is first established that the " white " numbers behave in the same way as the signless numbers of Arithmetic, then that the red numbers behave in certain cases as if they were " white " numbers prefixed by the opposite sign, and so on till all the various cases have been established. The rules for brackets can then be taken and for this purpose arithmetical examples which can be looked at in two ways are useful.

It will be noticed that the object of this work is not so much to give a proof for operations with directed numbers as to endue them with a purpose. The question " Why? " which is answered (or is it evaded?) by such work is not the " why " of the logician, but the " why " which means " To what end ". Care, therefore, should be taken to use the ideas of directed numbers freely in problems—for too often they are stored away as idle and meaningless pieces of manipulation. The ideal place for such discussion really comes later, when problems on quadratic equations are studied; for frequently rejected answers to problems may be discussed and the problem rephrased so as to permit their inclusion. But it is dangerous to leave so important a matter for so long and quite simple applications will bring home the usefulness of the new ideas. [*e.g.* In 10 years' time Mr. Jones will be double his son's age. In how many years will he be six times as old? The top mark was 80, the bottom 20; these were scaled from 0-100. Brown did his paper at home and sent it by post. He only got 15. How do I grade him? etc.]

Multiplication of directed numbers by signless numbers will probably have occurred in the course of this work and presents no difficulty. Here, again, in deciding how far to go in explaining the difference between $3 \times (+4)$ and $(+3) \times (+4)$ the teacher must be guided by the intelligence of his class. This is difficult and confusing for many boys and should, I think, be glossed over. For the treatment of the multiplication and division of directed numbers, it is necessary to make use of two independent scales of numbers. It is almost inevitable that time should be one of the quantities involved, in spite of the fact that this is one of the cases which involve the most difficulty with the movable zero. Possibly most teachers (especially those in the forties) will not want to speak much of some of the aspects of " zero time ", but some preliminary patter is inevitable. The second number is usually associated with some rate, *e.g.* a constant velocity or rate of change of temperature. By

considering cases in which the result is apparent a suggested general law is formed. In some way or other, the principle of continuity is invoked and then by suitable change in the data the general rules may be brought out. Division may be treated by deduction from multiplication.

VII. The work done in the establishment of the rules for operations with directed numbers is evidently allied to another phase of mathematical work which is usually considered a part of Elementary Algebra, viz. Graphs.

The idea of a graph as a mere pictorial representation of size is properly an arithmetical notion but the connection between two associated number-scales is clearly allied to the work we have considered above. It does not seem to be very material whether we take graphs before or after directed numbers. If we take graphs first, we have in hand a powerful method of illustrating the principle of continuity which, as we have seen, figures in the work on multiplication of directed numbers. Whereas if we do directed numbers first, we shall have a greater store of examples and illustrations for use in our graphical work.

The beginnings of graphical work lead to the last fundamental idea which may be regarded as part of Elementary Algebra—that of functionality. But it is a mistake into which it is easy to fall to introduce this notion too soon. In the early stages the graph should be nothing more than a pictorial representation of the formula—representation being via the height of the column. A good deal of practice with column graphs, including some discussion of the possible meaning and existence of intermediate columns is desirable before passing on to the next stage—which has been called the locus-graph. Here only the heads of the column are marked or plotted and indiscriminate joining of the points should be discouraged, or at any rate an adequate defence should be demanded first.

At this early stage, a most rigid insistence on "graphical good manners" is essential. The following list of points in graphical etiquette may perhaps be useful:

(*a*) Every graph should have a title, which should, if possible, be devoid of symbols.

(*b*) The axis should be marked with a reasonable scale and the name of the quantity to which the graduations refer. [1 inch to 7 units is never reasonable. 1 inch to 3 units only when degrees are being dealt with.]

(*c*) Observe strictly the conventions as to dependent and independent variables. *Never draw a left-handed graph.*

(*d*) Write in clear English any results, preferably on the graph.

It is not advisable at this stage to introduce the nomenclature of Analytical Geometry (axes of coordinates, abscissa, etc.). Graphs should be of "functions" or "expressions", not of "equations",

and it is best to adhere to this even at the risk of some circumlocution.

Graphs of functions and the consequent notions of functionality will be introduced better by consideration of practical examples than by consideration of certain fixed types of functionality. In particular, it seems very inadvisable to plunge into a discussion of linear functionality—this it would seem better to allow the class to find out for themselves. The ready-reckoner type of graph is useful for this stage of work *e.g.* a graph to convert degrees Fahrenheit to degrees Centigrade or to find the square roots of numbers.

I have no time to speak at length on the other parts of the Elementary syllabus—Factors, Fractions, Quadratic Equations, and so on. Very little, in fact, can be said of these parts which has not been said already. As I have mentioned, a practical approach to problems of notation and formulae is desirable and in providing material for this approach, other branches of mathematics may be freely used. When such subjects as literal equations and changing the subject of a formula are being studied, formulae taken from pupils' diaries and other such publications will often serve as a valuable text. And at the stage when examinations begin to play a part in determining the course, some revision and drill work is necessary and desirable.

Mention of examinations brings me to my last heading. It would be idle not to mention the lack of correspondence between the syllabuses of the various examining bodies. The coordination of these syllabuses is, in fact, one of the most pressing reforms needed in the sphere of elementary mathematics. We cannot pass on to those changes in the later teaching of the subject which I know the next two speakers are going to advocate without a clear understanding on what the elementary syllabus really is. At present, this is by no means clear. In one examination, elementary algebra ends with quadratic equations (of course, later parts of the subject occur in the Additional Mathematics papers, as they should). A second examination, very widely used, contains in its syllabus, in addition, arithmetic and geometric progressions and questions on negative and fractional indices. A third examination contains, in addition to all these, Ratio and Proportion, Variation, and the "theory of logarithms". Yet another examining body demands, in addition, the "theory of quadratic equations", but drops out progressions.

Here is an obvious place for reform and an obvious place to appeal for it. From experience in marking the scripts of some of these examinations, I am convinced that very few pupils gain any benefit from a study of the later parts of the subject, a study which absorbs time which would well be used in other and more profitable directions. He would be a hardy partisan indeed who could claim much general educational value for the theory of quadratic equations, even if it were granted that the majority of school certificate candidates can understand this subject—which is not the case. In fact, the vast majority of candidates who pass examinations, where such

topics are asked, do so in spite of the existence of questions on them or by avoiding them in favour of alternatives where it is permissible to do so. It is high time that something should be done to clear up this muddle and, as I have said, it is going to be very difficult to get any concerted action in the matter of Post-Certificate Algebra until we know clearly where that begins.

In considering the content of the ideal algebra syllabus, no question of the claims of intending mathematical specialists need be considered, for all these examinations have some form of "Additional Mathematics" which may be taken (and should be taken) by those who go a little faster in the subject. This is evidently the place for some of the topics I have enumerated and some consideration could well be given to the context of this course, as well as that of the Elementary course. Here the needs of the universities and of other types of post-certificate student (*e.g.* the intending actuary) could be carefully adjusted. It is along these lines that the most useful solution of the present difficulties might well be worked out and I should be glad to see some consideration given to both phases of this problem. We should then be in a far better position to consider reforms at a higher stage.

**Mr. M. H. A. Newman** (St. John's College, Cambridge) :

THE COURSE AS SEEN FROM THE UNIVERSITY.

Although I am to speak "from the University point of view" I wish to consider not only how the small number of boys who will go to a University may best be prepared for their work there, but also the general problem of the teaching of Algebra in the upper forms of schools. This problem is connected with a larger question, whether the balance of subjects is right in present-day school teaching in mathematics, and perhaps you will permit me to start with a short digression on this point.

It has been strongly urged recently that for many boys the grind of elementary mathematics, and especially the elements of Algebra, is a totally profitless one, since they never reach the stage where they can see something valuable or interesting emerging which would reward their earlier drudgery. This was the line taken by the successful attacks on the ancient languages as compulsory subjects, and it can hardly be denied that when transferred to mathematics it still has a great deal of force. One way to improve matters is to stress, from the very beginnings of Algebra, the underlying meanings and theoretical points, and recent numbers of the *Mathematical Gazette* shew the great amount of thought that is being given to this side. But in spite of all these improvements it must be admitted that there are large tracts of the elements of Algebra which are simply practising a technique.

The proper way to justify the presence of mathematics in the school syllabus is evidently to set beside the systematic drilling in

techniques some introduction to domains of mathematics that have an undeniable educational value. The way to introduce schoolboys to these higher reaches of mathematics is not, I think, to attempt to cover systematically the earliest parts of a University course on the subject, but to pick out suitable topics for very thorough and leisurely treatment, if necessary filling in the gaps between the selected parts by stating theorems without proofs.

The subject on which the choice has hitherto fallen for this kind of mathematical teaching is in most schools the rigorous treatment of the early parts of analysis—the " theory of $\epsilon$ ". There are good historical reasons for this choice, but it is in many ways an unfortunate one. Boys who make their first acquaintance with strict mathematical argument (on the analytical and algebraical side) in this way are led to think of " rigour " as something that is stuck on to a proof which is already satisfactory from a common-sense point of view. I am sure there are many Cambridge undergraduates who believe that " rigour " is a separate branch of mathematics, to be added to taste to other branches by those who have a fancy for it. Besides this, the manipulative work which is for many boys the attractive side of mathematics drops out in this kind of analysis, and it is liable to be supposed that " higher mathematics " consists entirely of abstract logical reasoning about familiar results.

In Algebra we have a subject which appears to be free from all these objections, and in every way suited to provide an introduction to genuine mathematical argument. There is no antithesis between " rigorous " and other proofs, and there is a sufficient amount of manipulative work to supplement the interest in the theoretical side.

Before going further I should perhaps say what kind of Algebra I have in mind. In school books of the nineteenth century the meaning of the word was gradually widened until it came to cover a most miscellaneous collection of fragments, including a good deal of what is now called analysis, but I believe that for professional mathematicians it has always retained its original meaning : the study of the " four rules ", addition, subtraction, multiplication, and division, together with the theory of polynomial equations. The two main divisions of modern Algebra (which have, of course, many interconnections) are *Linear Algebra*, including the theory of linear equations, matrices, determinants, linear dependence, quadratic forms, etc. ; and the *Theory of Equations*, leading on the one hand to the theory of invariants of binary and other forms, now somewhat out of fashion, and on the other to the theory of rings and fields, and of abstract number systems. It is from the first of these great divisions, linear algebra, that I think the material for an introductory course could be selected. The theory of equations side seems less suitable : without a background of invariant theory or Galois theory, which could not be attempted in a school course, methods of solving cubics and quartics by radicals have little significance, and are not even useful for numerical purposes.

In order not to leave the proposal for a linear algebra course too much in the air I should like to suggest, very tentatively, the outline of a possible programme. It assumes nothing but the absolute rudiments of algebra (what I believe is called " up to quadratic equations "), but does require familiarity with the notions of plane coordinate geometry (rectangular coordinates), and the equation of the straight line : it is now generally agreed that these ideas should enter the mathematical course at a very early stage. Although topics are introduced which ten years ago were considered " advanced " even in most universities, I believe that if the geometrical interpretations were kept well in view such a course would be intelligible to any boy, with the necessary knowledge, who is reasonably good at mathematics—certainly to many who would never make anything of " $\epsilon$-analysis ".

One would start with the linear transformation

$$x' = ax + by,$$
$$y' = cx + dy, \quad \text{(all numbers are real)},$$

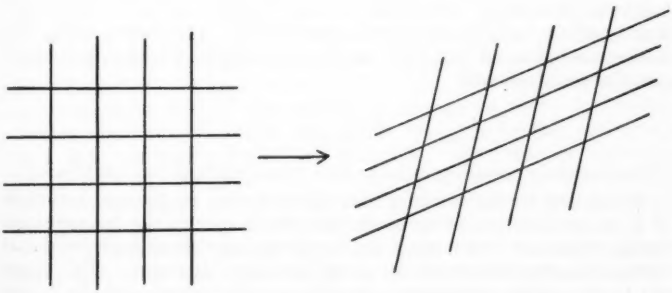
interpreting $x'$ and $y'$ not as new coordinates of the same point $(x, y)$, but as a new position in the old coordinate-system ; this avoids the use of oblique systems. The transformation is therefore a movement plus distortion of the plane. After a full geometrical examination of such special cases as

$$x' = y, \qquad x' = ax, \qquad x' = -x,$$
$$y' = -x, \qquad y' = by, \qquad y' = y,$$

and the " shearing movement "

$$x' = x + ay,$$
$$y' = y,$$

the general transformation would be interpreted as the closing of a



square lattice, with stretching of the arms, and possibly a mirror reflexion in addition.

A linear transformation is determined by four numbers, and the 2-rowed square matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is introduced as a scheme for specifying a transformation. The combination of two successive transformations leads to the multiplication of matrices, and superposition to their addition.

One would now do a certain amount of work on the algebra of 2-rowed matrices, shewing that all the laws of ordinary algebra, for $+$, $-$, and $\times$, hold except the commutative law of multiplication, and that the matrix

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

has the properties of a unit. As an example one could consider matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

If

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

then $i^2 = -1$ and

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a \,.\, 1 + b \,.\, i.$$

This leads naturally to a model for the complex numbers in terms of real. The transformation determined by the matrix $i$ is a rotation through a right angle, connecting up the model with the usual geometrical interpretation of $\sqrt{-1}$. The connection of De Moivre's Theorem with the $n$th power of the matrix

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

might be discussed. One might even go further, and shew how other less familiar "algebras" obtain consistency certificates, using the real quaternions $x1 + y\alpha + z\beta + t\gamma$ as an example, where $x$, $y$, $z$ and $t$ are real numbers and

$$\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(This involves stepping outside the framework of real matrices.)

Returning to the transformations, it would be pointed out that if a transformation is to cover the whole plane, the transformed lattice must not "shut up", *i.e.* the image-vectors of horizontal and vertical vectors must not lie along the same direction. This leads (*a*) to the notion of linear independence of vectors, and (*b*) to the number $ad - bc$, the determinant ( = the area of the transformed cell), whose vanishing is the condition for the transformation to be singular.

A real understanding of all the points raised in such a course would, I think, give a genuine insight into the nature of linear algebra. If it is thought advisable to go further the next step would be up to three dimensions. Here neither determinants nor the elements of coordinate geometry (direction cosines, etc.) should be assumed, but should be developed as part of the course. The final point to be aimed at would be the theory of the rank of a 3-rowed matrix, and of three simultaneous linear equations with vanishing determinant, interpreted in terms of the intersections of planes in 3-space.

I do not think matrices or determinants of order higher than three should enter into a school course, and there does not seem to be anything either in geometry or in dynamics requiring such determinants that could not very well be postponed. The full-blown suffix technique might also be reserved for University teaching.

**Mr. W. J. Langford** (Bideford Grammar School):

THE TEACHING OF ALGEBRA IN THE ADVANCED FORMS OF SCHOOLS

In 1934, the Board of Education asked the Mathematical Association to submit its views on the mathematical aspects of the organisation and interrelation of schools. In its report, the sub-committee appointed for this purpose, of which I was a member, made it clear that at present the mathematics up to School Certificate standard in schools of secondary education was inadequate in its scope and its direction for all but those pupils who will continue to read mathematics in a university course. If we remember that probably only some 10% of the population of our public and secondary schools passes on to any form of university career, and if we agree with the opinion of the sub-committee, we have before us a problem vast in its importance and extent and one which should be tackled without delay.

Perhaps I should explain that, broadly speaking, the School Certificate stage extends over some 4 or 5 years bringing the pupil to this examination at approximately 15 or 16. The Sixth Form course covers another 2 or 3 years of a more specialised study in Classics, Modern studies or Science and brings the pupil to something approaching university Exhibition standard. The object of the course is usually a Higher Certificate or a university Scholarship or both. Mathematics plays an essential part in the School Certificate and may also enter into all the Sixth Form courses.

The criticism of present-day standards by the sub-committee referred to was that our courses as conceived at present are too academic and this criticism may well be extended to the advanced course. They are adequate and wholly desirable for the few—perhaps some 1%, at most, of the school population—who will read mathematics at the university but do not pay sufficient regard

to the claims of the rest, a very large and important majority. Certain suggestions were put forward in the report for remedying this state of affairs amongst which was one which I regard as of the utmost importance, *i.e.* that those responsible for the various stages in the mathematical education of the individual should work in the closest touch, one with another. Though the needs of those who leave school for business or industry cannot be a direct concern of the universities, yet if we agree that their claims must be heard, by reason of their number and the fact that they are co-equal citizens with us all, then it should be possible by a frank interchange of view between those responsible for the mathematical teaching in the universities and the schools so to arrange the affairs of each that what was adequate for the vast majority who do not continue their mathematical education beyond the School or Higher Certificate stage should at the same time provide a sound foundation on which to build a university course. I am willing to agree at once that the problem is a big one and that there must be a willingness to give and take by both parties in the discussion but up to the present the "rest" have had to give all and I for one feel that it is time those of us who teach mathematics in the schools should begin to urge a case on their behalf. This gathering to-day provides a unique opportunity to begin this work and I am deeply grateful for the privilege which is given me to speak to you from this point of view on the teaching of Algebra.

You will have seen already how great a change the last 30 years has wrought in the teaching of the essential groundwork of the subject. I wish to give you some indication of the effect of these changes at the School and Higher certificate stages together with some suggestions for further change so that we may link it all with the views just put forward so admirably by Mr. Newman. We must remember that some five-sixths of our school population (it may even be more) does not continue any form of mathematical education after the age of 16 or 17. With reference to their future therefore, some care should be exercised to see that the equipment they have should be reasonably adequate, inspiring and useful. The theory of the transfer of training seems now to be completely exploded and while Art for Art's sake is still a good text to preach from, yet it seems that if without sacrificing these ideal standards too much we can at the same time ensure that the subject has some utility and is moreover a sure foundation on which the individual may, in that leisure which we hear so much about to-day, continue to build for both pleasure and profit—if we can ensure this—we are surely approaching that standard we set ourselves a little earlier.

The master responsible for the organisation of the mathematics in a school (I refer to the masculine in this paper—I believe the feminine problem is much the same) has consequently a multitude of claims to balance when deciding the scope of the work in the school. Examination requirements at once limit the freedom of his choice and as long as group requirements exist so long is there the

risk that algebra at this stage may remain a technique spread over a few isolated and somewhat barren topics of the subject. This need not be so, but the stress of modern conditions often restricts the teaching to a bare minimum. In the large schools, by the operation of sets, it is possible to do much to ease this burden but in the smaller one-stream schools good, bad and indifferent must work together and the scope is thereby much reduced. Let me say at once that it is possible for the schools to submit special syllabuses to the examining authorities and it seems that by a solid advance along these lines, much improvement can be effected. In many cases, the boy leaves his algebra at the School Certificate stage, often without knowing anything about the Binomial expansion, and has thus obtained a technique which is sterile. He cannot, except in rare instances, continue his mathematical education without expert guidance. The boy who leaves his French or Latin in like circumstances, should be able, with the help of a dictionary, to enjoy the literature of the language. In Science, the study may still go on unless mathematics plays an important part. Once again we see the need for a revision of outlook. Herein lies the germ of what we are discussing to-day. What should we teach in algebra and how should we teach it in order that *all* classes of the pupils in our schools should reap the benefit from the work at whatever stage they leave it. Clearly an ideal ; but it does cut severely across that point of view which says " Here is an examination ; pass it and you may go out into the world an educated man. Fail it and—well, comfort yourself by reading *An Examination of Examinations.*"

Present-day methods in the teaching of elementary algebra have tended to react strongly to this need for a less theoretical attitude towards algebra as a school subject. Mr. Parsons gave you ample evidence of this in his paper on the treatment of the foundations of the subject. It is also very noticeable in the treatment of equations and series. We force the boy to create a quadratic equation through a well chosen problem : then, having it before him, he has to confess that he cannot solve it with the means at his command. We then must try to get hints for a method from the boys themselves. In teaching, as distinct from lecturing, one is so dependent on the initiative and suggestion coming from the boys themselves and it needs a very dull group indeed to be devoid of useful suggestion when the ground is carefully prepared by skilful leading questions.

I cannot urge too strongly the need for some early training in tabulation. All sorts of interesting applications arise as the work progresses and it is better to have the method available early than to be forced to introduce it just as it is required. Further, the really young mind is very willing to accept algebraic manipulation as a game to be played for its own sake and a good deal of formal drill can be done early provided the examples are easy. Boys at the age of 12+ and 13+ will really enjoy working 8 or 10 examples almost identically alike, whereas at 15+ or so the fact that they are so nearly alike breeds a boredom which defeats our purpose. Algebra

up to the School Certificate stage consists almost entirely in gaining a technique for use in other studies, many of which at present never enter the scope of this general education even by passing reference. This emphasises a need for the teacher to introduce as many applications as he can secure. There should be as much unity as possible in the work at this stage and we must ensure that we make our topics as complete as possible and at the same time link up the work in hand with as many interests as we can. It is better to derive our equation from the activities of a man running to catch a bus and then having found the equation to discuss the meaning of the solutions, especially those which may appear invalid, rather than the approach which used to say—" Here is an equation, called a quadratic equation. To solve it proceed thus : "—and then in some cases gave even the method of completing the square before the method of factors.

In dealing with series, the same attitude has produced similar changes. Instead of treating only the cases of the A.P. and G.P. and, moreover, regarding them merely as a convenient means of introducing formulae to be memorised or proved we now find endless variety and interest in sequences long before any idea of summation is thought of at all. The psychologist is fond of using sets of numbers as tests of intelligence at quite early ages. When sequences and series are adequately dealt with at an elementary level, the cases of the A.P. and G.P. become almost trivial extensions of a really important part of our algebra and the general question of finite series can be easily added when occasion arises. The need for a method of summation is best brought out by means of a problem and once the need is realised, the value of this economy of effort is appreciated and gives point to a method which is valuable only by reason of its applications. The sum to infinity seems best avoided altogether as the idea cannot be fully appreciated at this stage. It is, however, worth while to discover how many terms of a sequence must be taken before the sum exceeds a certain quantity. Manipulation with inequalities is all much neglected nowadays and in the application of this discussion to the G.P. we have a number of valuable lessons to be learned as the calculation proceeds, matters which develop naturally and so give purpose to the investigation. If we set out to determine how many terms of the G.P. $1+\frac{1}{2}+\frac{1}{4}+\ldots$ are needed before the partial sum exceeds 1·999 (say) we reach the equation

$$2-(\cdot 5)^{n-1} > 1\cdot 999$$
$$(\cdot 5)^{n-1} < 0\cdot 001$$

Now this raises the solution of an equation with the unknown as an exponent. Having decided on the method, perhaps after a digression, we find that the unknown has contrived to get to the wrong side of the inequality and we are able to rub in the point of the negative characteristic of the logarithm. In a report published in 1932 by a panel of investigators appointed by the Secondary Schools

Examination Council, the question was asked whether the progressions were worth doing at this stage and the opinion expressed that the time might more profitably be given to easy calculus. Our answer now seems clear. The progressions are not worth doing—but sequences and series into which these progressions fit are abundantly valuable as a training ground in technique and as a means of uniting other topics, which might well remain isolated ideas. Throughout our algebra teaching, we should consciously be teaching mathematics. This may sound trite, but all who have taught in schools will know only too well how hard it is to make the boy think of his geometry when he has made up his mind that the lesson is algebra. What I imply is that geometry and trigonometry and the fundamental ideas in the calculus and coordinate geometry must continually be brought into our work possibly before any mention has been made of these as separate studies. A graphical treatment of the partial sums of the series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ not only helps the young mind to grasp the ideas of the algebra but also provokes the boy sooner or later to ask the awkward questions that we, as teachers, long for. In this case, the boy is led without knowing it to the concept of convergence and to the elementary idea of an asymptote.

So throughout the work in the School Certificate stage we must strive for unity and completeness in our algebra teaching. We must regard factorising, the simplification of algebraic expressions and the other formal work merely as an instrument, a technique to be used for specific purposes of real value. Let the boy see at once that factors solve his equations, that equations solve his problems and amongst these let there be problems which are only partly solved when the value of $x$ is known. Drill, of course, there must be, but it should come after the use of the method has been clearly appreciated and it must not be overdone. The so-called harder factors have no place here. Their treatment results only in a technical mastery which produces no results at this stage. There are also other isolated topics which might well be cast away at the same time. The Factor Theorem, however, is of first importance as is also all the work with formulae. Neither should it be possible for a boy to leave school without knowing at least as much of the Binomial expansion as can be brought out in a short lesson on Pascal's triangle. Graphical work could and should be extended so that all the ideas of gradient, maximum and minimum and limits are there for the development of the calculus as soon as they are needed. I have known boys at this stage tackle quite complicated graphs of rational functions and find real pleasure in their ability to discuss the changes in the value of the function. They are then not far from the conception of functionality and continuity can be developed intuitively as soon as we please.

When we come to the Sixth Form course, the need for extreme care in deciding the scope of the mathematical teaching is emphasised very strongly. In this course, lasting 2 or 3 years, the boy begins to specialise and it seems that we should be able to make the

subject live for its own sake, as well as for its use in other studies. Nevertheless, we find here a multitude of varying needs calling for a different scope, a different treatment or a different attitude in our mathematical teaching. In addition to the specialist, there may be, in the same group for teaching, boys who intend to read Natural Science, to become actuaries, surveyors or engineers as well as those who are taking additional mathematics to make up an Intermediate qualification in Modern studies or Classics. Quite definitely the needs of the specialist are not best suited as a guide to the course to be given to all, though it must be admitted that at present the claims of those who will take mathematics as a subject in open scholarship or Higher Certificate are so far-reaching that for justice to be done here, the claims of the rest must be largely disregarded. In the Sixth, the boy who reads mathematics at all seriously will be prepared to accept some algebra for its own sake and not merely for its immediate use elsewhere. Unfortunately the ever-present factor of time enters largely into the problem. In the larger schools, it is possible for the first and second years to have separate teaching programmes and to receive something like 10 hours a week to cover both the pure mathematics and the mechanics. In the small schools, all the boys taking mathematics must go together in the one teaching allowance, which is often little more than half of that in the larger schools. Staffing will not allow a more generous allowance of time and it is rare that there will be two men on the staff of 10 or so able to deal with the work. To this we add the much extended syllabus of work and the prior claims of those branches which have so far received scant treatment—calculus and coordinate geometry and possibly mechanics—and the algebra takes a very lowly place. At present, only the bare outlines of the work can be dealt with. This is in strange contrast to the ultimate importance of algebra as a key subject in our mathematical equipment. An adequate training in manipulation is essential for any comprehensive study of mathematics, and I feel strongly that a good deal of the difficulty which boys experience in mechanics is due to their failure to deal with the equations reached rather than a failure to appreciate the mechanics, and in the algebra of the Sixth Form we should find ample opportunity to introduce the fundamental features of the rigour of a proof; the consideration of all the possible conditions and cases with the gradually increasing array of consequences to be sifted before the exact statement of the proof can be formulated. This work cannot be attempted to any extent at the School Certificate stage but should come as early as possible in the advanced course where the treatment must of necessity be more rigorous, for here mathematics is being studied for itself alone and not as a part of a general education. The schools must set themselves to care for the mathematical education of the whole group studying the subject, not necessarily at the expense of the specialist but by altering his work in such a way that while it provides him with a sound foundation on which his university course can be built, we still give the

rest a comprehensive course which is in some ways complete in itself and makes it possible for the subject to be pursued with a different emphasis if the need arises. A good deal of the geometry and trigonometry of the triangle and quadrilateral should go at once. We must teach coordinate geometry and not conic sections. We could well omit all the analysis which depends on the use of $\epsilon$; the ordinary sixth-former does not profit at all from following such proofs and his time could be more profitably employed in furthering his study of algebra. As for the specialist, there is surely time in his first year at the university to tackle these essential foundations of his work. I am firmly of the opinion that the specialist can reach the university every bit as well prepared to study mathematics, even though almost all the analysis is unattempted, provided reasonable care is exercised in planning his work so that his advanced course at school fitted smoothly into the first year study at the university. I am not implying that the schools cannot be trusted to teach this work; they can and they do, but I feel that it is at a price which they should not be called upon to pay.

With regard to the content of the Sixth Form work in algebra, I think that an attempt should be made to introduce early the Hardy treatment of the study of the complex number. In one vital piece of work we unite the hitherto independent processes of algebra and trigonometry, we resolve the difficulty of the so-called impossible solution of the quadratic equation, we close the field of operations with the processes of arithmetic and above all we give a hint concerning the possibilities of axiomatic mathematics. Except perhaps for a digression on the consequences of the failure of the 5th postulate the boy so far has believed his mathematics real in the popular sense; to be an abstraction from the concrete. He has no ideas regarding pure theory—no hint that there may be other algebras and geometries well worth studying for their own sakes, quite apart from the ultimate results which these studies give to us. The treatment of the complex number as an ordered pair immediately gives rise to the use of vectors, opens a new approach to some elementary geometrical ideas and provides the boy with a most useful tool for use in mechanics. Moreover, the boy's mind at this stage can well appreciate the work. There are no deep philosophical difficulties, merely a new method of approach coupled with some familiar deduction. The boy, even if he goes no further than the 2-year course, does gain a much greater idea of the power and scope of mathematics. Again, a careful treatment of determinants and matrices, such as that outlined by Mr. Newman, is easily possible in the Sixth Form if only the time can be found at the expense of some other less valuable parts of the work we are now expected to do. When I say less valuable, I mean this, of course, from the point of view of the teaching and not with reference to the importance of the work itself. There is nothing in my experience quite so vivid as the eagerness with which a boy sets out on his first solution of a set of simultaneous equations by means of determinants. The sense of his

power in being able to side-track the old grind of step-by-step elimination inspires an enthusiasm which makes the teaching well worth while. We should find room for such pieces of work which offer us as teachers much that is of value and yet present conditions often prevent the non-specialist from having any acquaintance with this part of algebra, other than perhaps the use of the determinant notation as a shorthand symbol in the formula for the area of a triangle.

We are then face to face with another problem. How are we to proceed to bring about these changes? It would be dangerous for the schools to alter the scope of the teaching, unless the examinations move in the same way. Much as we may dislike the tyranny of examinations, we must agree that School and Higher Certificates now constitute a very necessary label in the search for employment. Neither will it do for the examinations to change and the teaching remain, as of old, a sequence of isolated topics and bits of technique. The same failure would result, though I am inclined to believe that the new teaching would bring better results even if there were no considerable change in the examinations. One way is for teachers to agree upon a policy and in the light of it submit alternative syllabuses. The second is the formation of a Committee of Contact between the examining authorities and the teachers in the schools. The second I feel is by far the more preferable method. This would bring about a change more rapidly than any conference could expect to do. For the university scholarship, the problem is not so simple. Here the algebra receives usually less treatment than in the Higher Certificate and the emphasis is more strongly upon the side of analysis, for the scope of the calculus is extending in such a way that, to deal with it at all adequately, much of the elementary function theory must be covered. I am often tempted when reading through scholarship papers to compile a list of the essential results which must be available in a candidate's equipment for the answer to one question to be founded upon a sound mathematical knowledge rather than upon a chance selection of topics by the master. The list is often illuminating. I am far from suggesting that we should not teach much of the calculus—rather would I urge that it should be started much earlier in schools than is now usually the case—but I am of the opinion that we could usefully restrict its scope in the directions which touch analysis closely and so make way for other parts of algebra which should claim some attention in the schools both for their own sakes and for the solution of the problem as to the course which will give the varying claims of the boys taking it something like equal treatment.

In order to bring about such a change as this, nothing will suffice but an agreed policy on the part of the universities and the schools. Since both are concerned with teaching, quite apart from the function of examining, the personal contact could and should be used. If Mr. Newman may be regarded as an ambassador not extraordinary the search for a solution should not be a lengthy one.

I confess that I have dealt largely with generalities and that much of what I have said can be applied directly to all the mathematical teaching as well as to the study of algebra. If, however, I have been able to provoke some reaction either for or against my point of view and so give rise to a useful discussion I shall have no regrets.

It remains only for me to say how much I appreciate this opportunity to speak before you and how glad I am to have done so under the chairmanship of Professor Neville to whose inspiration, guidance and personal friendship I owe so much.

## THE BEARING OF HIGHER GEOMETRY ON THE SCHOOL COURSE.

*7th September*, 1937

THE Chairman, Professor H. T. H. PIAGGIO (University College, Nottingham): The subject of discussion this morning is "The bearing of higher geometry on the school course". The difficulties of geometry are well-known, though not all of you will entirely endorse the opinion expressed about 350 years ago by a certain Hubert Languet, who wrote to Sir Philip Sidney as follows: "I know not whether it is wise to apply your mind to geometry, though it is a noble study and well worthy of a fine understanding, but . . . you are not overcheerful by nature, and it is a study which will make you still more grave; and as it requires the strongest application of the mind, it is likely to wear out the powers of the intellect, and very much to impair the health."

Without taking this extreme view, there are certainly some grounds for suspecting that pure geometry offers special difficulties which do not occur in other branches of mathematics. I can recall a large number of students who took first class honours in mathematics, who, nevertheless, could make little or nothing of projective geometry. In the language of the psychologists, geometry seems to need what is called a special ability, in addition to general intelligence and general mathematical ability. What are these special mental powers that geometers possess? Unfriendly critics may say that the first requisite is faith, the power to believe what we know not to be true. Those so endowed can unblushingly deduce beautiful and even true theorems from such outrageous nonsense as $1=0$, or from the properties of lines perpendicular to themselves. They appear to believe, in the words of Mr. Hope-Jones, that "infinity is the place where things happen that don't!" But, as later speakers will explain to you in detail, this childish nonsense can now be discarded. Unlike Peter Pan, geometry *can* grow up; unlike Tinker Bell, its life is *not* dependent on a belief in fairies! Personally, I do not think that either logic or the absence of it is the leading characteristic of a geometer's mind. Projective geometry may or may not be merely concealed analysis, but one's judgment on that point does not help or hinder the exercise of geometrical insight. When I contemplate the minds of my geometrical colleagues, as I do with admiration and awe, what strikes me most is their power of visualisation. They see a diagram in their mind's eye, and immediately recall the relevant theorem, while those of us who are less gifted, and cannot rely upon the inspiration that will take us straight to the heart of the problem, are compelled to set the analytical machinery in motion and grind out the result required.

In the discussion that follows, the term *school course* must, I think, be interpreted in a rather restricted sense, as the course of preparation for scholarships at Cambridge or Oxford. Professor

Neville will consider how far it is profitable for schools to anticipate university work in geometry. Then Mr. Green and Professor McCrea will outline a rational treatment of the line at infinity and the circular points at infinity, and Professor Ruse will plead for a more extended treatment of elementary differential geometry.

**Professor E. H. Neville** (University of Reading):

THE INFLUENCE OF THE UNIVERSITY ON SCHOOL GEOMETRY

In opening this discussion on the influence of the university on the school course in geometry I have two dangers to avoid. Many of the things which it would seem natural to say are true and a few of them may be important, but when I wrote them down and took a second look at them, I saw that they were not concerned in any specific way with geometry; we sometimes forget that there is a Section of the British Association which exists to rescue the rest of us from the generalities and platitudes to which teachers are only too prone. The other danger is of entering so far into particulars as to compose a syllabus round which we might be expected to quarrel instead of talking about principles on which we can hope to agree; here I am fortunate in the subjects chosen by the speakers who are to follow me, for where what I have to say is most in need of illustration, their expositions will provide you with examples worked out at the appropriate length.

Geometry is a subject in which, at the time when the coming university casts its shadow before, manipulation and ideas can be separated to a remarkable extent. What I have to say amounts to insisting that to press forward with manipulation without giving time to ideas is an injustice both to the student and to the subject, but that we need not be afraid, especially with students to whom mathematics is only part of a general education, to devote a little time to ideas dissociated from manipulation.

Except in trivial detail, the subject matter of elementary geometry is hardly open to discussion. When we take notice of the shapes of the objects around us, and attempt to extract forms and relations which are simple enough to reason about, inevitably we deal with congruence and similarity, with angles and parallel lines, with right-angled triangles and the consequences and extensions of Pythagoras' theorem, with the circle and its chords and tangents, with incidence properties of lines and planes in space, with the less intricate polyhedra, and with the sphere. Throughout this period, our results have a fairly direct practical bearing, even if it is only as the theoretical basis of practical constructions. But practical considerations become gradually less and less relevant, and to some temperaments more and more tiresome, and when the wonder of the angle property of the circle has been followed by the wild surprise of Feuerbach's theorem, it is a sadly unresponsive or incurious learner who is not in a mood to study geometry for its own sake. No longer

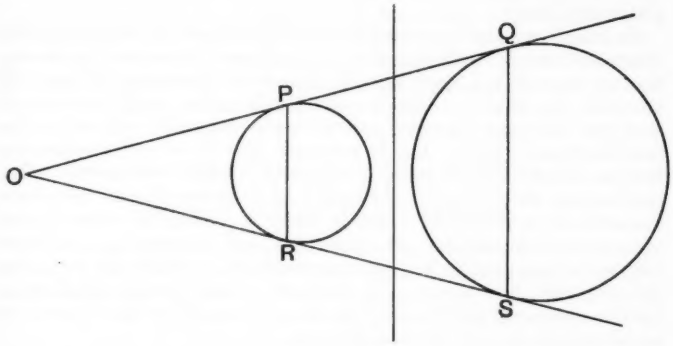
is it taken for granted that if we just keep our eyes open our minds will be satisfied; henceforth, it is on the mathematician that we rely for the selection of subject matter. I invite you to consider some of the principles that should govern the selection, and to examine the subject matter to which we have grown accustomed, and I suggest that if we were not hampered by tradition, we could, perhaps after painful experiments, make subtractions and additions which would be to the benefit of everyone concerned.

Is it not fair to say that the practice in the past, with regard to higher geometry in schools, has been to press forward as fast and as far as possible with the manipulative developments of a few branches? In pure geometry, this has meant the use of harmonic ranges and pencils. The polar of a point for a circle is the locus of the fourth harmonic, or in other words, conjugate pairs of points on a line are pairs harmonic with the pair in which the line cuts the circle, and since almost the first theorem learnt about the harmonic relation between two pairs is that it is projective, to define the conic as the projection of a circle is to extend the whole theory of poles and polars automatically to the conic, and it is all so simple that presently we have the circular points in our picture if not in our diagrams and are projecting ordinary accessible real points into them. We actually say we are doing projective geometry when we produce this sort of thing.

There are difficulties too gross to be overlooked. At the outset, what if the line does not meet the circle? Well, we appeal grandly to a principle of continuity, which we probably attribute to Poncelet. We may not say in so many words that according to this principle whatever is true sometimes is true always, but I am sure that as a rule that is the meaning attributed to our language. Because two circles sometimes cut in two points, it is to be legitimate always to say that two circles cut in two points, until presently we find that because two ellipses sometimes cut in four points, it is to be legitimate always to say that two circles cut in four points.

In parenthesis, may I say that Poncelet never did maintain the unmitigated nonsense so often attached to his name. What you will find in his famous *Propriétés Projectives des Figures* is a very careful statement that if a property of a figure can be enunciated in such a way that no reference is made to points and lines that exist in some cases and not in others, then as a rule if the property is true when the points and lines exist, it is true also when they do not. For example, let two circles have two common tangents $PQ$, $RS$, of a prescribed type, that is, both external or both internal; we may express the fact that the radical axis bisects $PQ$ and $RS$ by saying that it is the line midway between the parallel chords $PR$, $QS$, but this form of enunciation fails if the common tangents of the prescribed type do not exist. Now let the common tangents cut in $O$; we recognise $O$ as a centre of similarity, and the chords $PR$, $QS$ as the polars of $O$ for the two circles, and *if* O *is a centre of similarity of two circles, the radical axis of the circles is the line midway between the two polars of*

O, *provided that there are common tangents intersecting in* O. The principle of continuity sees the last clause to be unnecessary, not




because in some sense there are common tangents when there are not, but because there is no reference to these tangents in the theorem as it is now stated.

To resume, it is only if the generality of the arguments can be maintained that any type of manipulation is efficient. Are we to accept work with harmonic ranges as meaningless drill in a language which will some day be given a meaning, and is familiarity with this adult language so important for our young mathematicians that it is better for them not to understand it than not to use it? Personally, I think not. It may be that the whole matter can be presented simply and briefly in a manner that is not an insult to their intelligence. If not, I contend that the subject is not suitable for school use, however important it may be at a later stage.

In analytical geometry, it is perhaps the line at infinity that has suffered most from premature introduction, usually by way of homogeneous coordinates. The absolute coordinates of every point satisfy an equation $ax+by+cz=1$, where $a$, $b$, $c$ are constants of the system; nevertheless we propose to say that assigned ratios $x:y:z$ always represent a point, even if $ax+by+cz=0$, and since the equation $lx+my+nz=0$ represents a line unless $l:m:n=a:b:c$, we propose to say that the points whose coordinates satisfy the impossible equation constitute a line, anomalous though that line may be. Again I ask whether we do any good to young mathematicians by insulting their intelligence to this extent. If we feel sure that homogeneous coordinates and the line at infinity are things they should be able to talk about, we must spare a little time from exercises in technique to give some rational account of what we are doing. The coordinate geometry of the university is a highly

sophisticated development from the naïve measurements by which a buried treasure is located, but we need not give the impression that we interpret the absurdity $1=0$ and colonise some region beyond the depths of space with points and lines whose existence we emphatically deny.

In imaginary and inaccessible points and lines we have examples where the use of words has often been allowed to outrun the formation of concepts; I hope that Mr. Green and Professor McCrea will persuade you that the schoolmaster has been too timid in this field and that adequate theories can be made plausible without undue consumption of time. I only reiterate that if we are to enter the field at all, we should do so intelligently. I turn now to my other contention, which is to be illustrated by Professor Ruse. There are branches of geometry in which it really is possible to impart some appreciative knowledge of important and interesting problems without attempting to describe the methods by which the problems are attacked. The words to be used have a more precise significance for the mathematician than for the man in the street, but there is no paradoxical language to be explained.

It would be stupid to pretend, for example, that to speak of a network of curves on a surface is to convey no meaning whatever to a schoolboy. We can point out that it is easy to reticulate a ring-shaped surface completely and we can affirm that in fact it is impossible to cover a sphere with a simple network although obviously we can get so far as to concentrate all our troubles at a single point; we deduce that there is some profound difference between the spherical surface as a whole and the annular surface as a whole which we could detect by trying to reticulate the surfaces even if we could not visualise them in space of three dimensions. To gain such glimpses into the actual problems of differential geometry is far better than to burden our textbooks on calculus with a section dealing as exhaustively as time permits with the curvature of plane curves. Similarly we could do something to present the idea of transformations in a better light. Are we not in danger of letting inversion and projection and reciprocation look like underhand ways of proving results of which we are not clever enough to discover the real foundations? Do we exhibit the coaxal system of circles as a linear system that can be rebuilt on any two of its members, or do we talk too much of the radical axis and the common points or limiting points as the case may be?

It is, of course, fatally easy to shirk drudgery and to spend exciting hours skimming the cream from other people's milk. But the embryonic mathematician has a future too rich to be impoverished by the most predatory schoolmaster, and he will undergo drudgery the more cheerfully if he sees something of the prospect ahead. Also it is only in descriptive talk of this kind that the teacher of mathematics can offer anything of value to the boy whose mathematical education is to stop when he leaves school, while it is no bad thing to let the future specialist in science or economics who accepts

readily the assurance that mathematics is a tool which he must learn to handle, see where mathematics can lead when it is pursued as an art.

An inspiring use of the ideas of higher mathematics is beyond the teacher whose library contains only schoolbooks and who is content if he can keep a chapter ahead of his class. I suspect that in the days when the whole curriculum in geometry consisted of Euclid's first four propositions, the successful teachers were those who had crossed the bridge of the fifth.

**Mr. H. Gwynedd Green** (University College, Nottingham):

### Infinity in Euclidean Geometry

It has been the fashion in some branches of mathematics to use a rather high sounding and mystifying terminology for comparatively simple ideas and I fear that Geometry has fallen to the lure. It is, of course, essential that new conceptions should have new titles. There is, however, the very serious danger that, through familiarity and, perhaps, moments of carelessness more may be read into such titles than was originally intended. I propose to discuss Infinity in Euclidean Geometry, as I find continually, especially with young students, that the phrases " point at infinity " and " line at infinity " have led to concepts of concrete existence which, whatever may be the case in other philosophies and branches of science, are in this one quite unjustified and which would be exceedingly awkward if they were correct.

In the first place, the pedant will at once object that within the strict limitations of Euclid there is no need for the concept in any form, and he will accuse me of an error of terminology and of falling into the traps at which I am making a general tilt. I must, of course, plead guilty, as I am thinking of the geometry which has developed naturally from the Euclidean scheme and for which, for want of a better description, we may borrow Casey's phrase " A sequel to Euclid ", but, in extenuation, the error was deliberate in order to exhibit the ease with which the trap may be sprung.

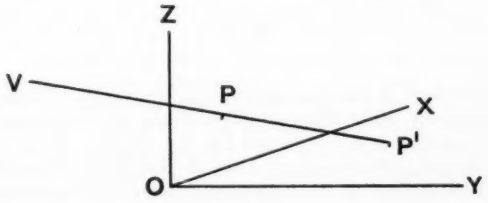



Fig. 1.

If we take two planes inclined at an angle and any point $V$ not on

them, the line joining $V$ to a point $P$ on one of them will usually meet the other at $P'$ : $P$ is perfectly definite, and vice-versa, if we start from $P'$, $P$ is perfectly definite : each is called the projection of the other from $V$. In terms of algebra, if $y$, $z$ are the coordinates of $P$ in its plane, those of $P'$ are $cy/(c-z)$, $az/(c-z)$ in its plane where $V$ is the point $(-a, 0, c)$. If $P$ is given various positions on a general straight line or a curve, the corresponding $P'$ takes up corresponding positions on a straight line or curve known as the projected figure : the intersections of curves lead, usually, to intersections and the degree of a projected curve equals that of the original curve. The construction fails, however, if $P$ lies on the straight line (algebraically $z=c$, $x=0$) for which $VP$ is parallel to the other plane : there is then no corresponding $P'$ and this process leading to its disappearance is called projection to infinity. Any other straight line projects into a definite straight line, but this one disappears : it is said to be projected into " the line of infinity " but there is no suggestion of any concrete point or line at infinity and all we know is that a point near this vanishing line projects into a point at a great distance. The corresponding process of projection " from infinity " is, I think, obvious and when in a geometrical problem we speak of a point at infinity in a figure, we are not thinking of that figure at all, but of a projected figure and of an extra point which has appeared in it. The terminology is merely an abbreviation summarising this process.

Let us now consider a point of intersection $A$ of a curve of degree $n$ with the vanishing line.

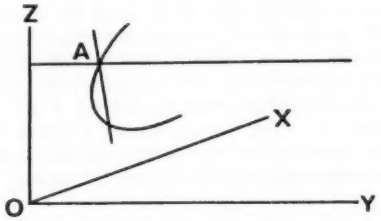



FIG. 2.

No line through $A$ cuts the curve in more than $n-1$ other points, and for simplicity, restricting ourselves to the simplest case as in the figure, the number is $n-1$ except for the tangent at $A$ when it is $n-2$. In the projected figure we have then a family of parallel lines, each of which is short of the normal number of intersections by one except for one of these lines, which we call the asymptote, which is short by two. Taking more complicated cases we are led to the general idea of asymptotes—" if a family of parallel straight lines are short

by $r$ of the normal number of the intersections of a straight line with a curve and if any one of them is short by a greater number then this one is called an asymptote ". Further, by considering the bearing of the tangent at $A$ to points near $A$ on a curve, we obtain an idea of the bearing of the asymptote to a projected curve at great distances. It is, however, in no way suggested that the asymptote is a tangent at infinity. I gave a full treatment from this point of view with the algebraic parallels in the *Mathematical Gazette* some years ago : it will be sufficient to point out here that in algebraic calculations on asymptotes we avoid arguments, which have always seemed to me unsatisfactory, of the type " we have obtained an equation in $x$ of degree say five—it isn't of degree five at all, we will tack on a $0 \cdot x^6$ and it is of degree six ". At the best it is artificial, at the worst it may sometimes be a little difficult to condemn much more generous additions.

The process of projection is not in general amenable to a discussion of metrical properties. There are, however, two important exceptions : one consists of the double ratio known as the anharmonic ratio and the other arises from pairs of lines $PA$, $Pa$ through any two selected points on the vanishing line. In the projected figure we have, for the first case, a persistence of the anharmonic ratio and, for the second, pairs of projected lines containing the angle $AVa$. One case is of special importance, namely the case when the angle $AVa$ is a right-angle which falls most naturally as a second type of exception, though actually it can be developed from either. As the pair of points $A$, $a$ adjust themselves along the vanishing line, they form what is known as an orthogonal involution vertex $V$. There are two schools of thought in the treatment of this involution. In the first the involution is treated straightforwardly without further convention—I believe one of my fellow speakers intends to develop it—in the second the idea of imaginary points is used as a basis, but in either case, after the first basic steps, the work is identical, so I leave the details to him. I must confess a tendency towards the second outlook although it admits algebraic ideas beyond the range either of Euclid or of its Caseyian sequel. By the second method we establish at once two imaginary points on the vanishing line such that all curves of the second degree through them give rise to circles on projection. As these points are on the vanishing line they give rise to the phrase " the circular points at infinity " after the projection. The phrase is convenient and is not dangerous if my basic premise of a reversion from one figure to the another is accepted—that is to say, that if instead of thinking of a circle through circular points at infinity one really thinks of a conic through two particular points on the vanishing line, and further there is no great difficulty in illustrating these two points as if they were real.

From these ideas of the line at infinity and circular points, derived from a projection in space from plane to plane, and keeping of course within our reservations as to the interpretation of the terms, we can develop to the plane at infinity and circle at infinity from a

projection in four dimensions from space to space. This extension is powerful and of value, but I will rest content with pointing out one application. From the quadrangle formed by the four points of intersection of two conics, we can obtain very rapidly, not only the elementary properties of the general conicoid such as the generators and sets of conjugate planes but also more difficult ones such as the circular sections and the focus-loci of parallel plane sections. Moreover, the full reasons for the modifications which arise in special cases are made apparent in cases where they are often left obscure in algebraic investigation.

**Professor W. H. McCrea** (Queen's University, Belfast) :

THE CIRCULAR POINTS AND ELEMENTARY GEOMETRY

Elementary geometry will be supposed to have its usual meaning as euclidean metrical real geometry of the plane. I propose to point out that, while " points at infinity " and " the line at infinity " may be properly introduced in conformity with the axioms, the " circular points " cannot be so introduced. They do not belong to this geometry ; and those results which do properly belong to it, and are often made apparently to depend on the existence of the circular points, should therefore be expressed with no mention of these points.

*Points at infinity and the line at infinity.* We shall first see, in outline, how these may be introduced with no modification or extension of the geometry. So we may assume all the known theorems of the geometry which do not involve these concepts. We shall show that their introduction may be regarded solely as a linguistic device for giving convenient expression to consequences of the parallel postulate. Theorems which may be labelled " projective " will be seen to hold for ordinary points and lines and also for points at infinity and the line at infinity, without any distinction for the latter, so that they may be stated without exceptional cases involving parallel lines. This, in turn, prepares the way for the study of pure projective geometry, and particularly for the introduction of the euclidean metric into this geometry. In fact, what we are here doing is almost the converse of the latter procedure.

" *Special points.*" We have, amongst others, the axioms :

($\alpha$) *Any two points determine a single line* containing them both.*

($\beta$) *Any two lines which are not parallel have a single point in common.*

($\gamma$) *Through any point not on a line* l, *there passes one and only one line* m *parallel to* l. *The lines* l, m *have no point in common.*

We shall now introduce a provisional nomenclature, so as not to anticipate matters. The points and lines of Euclid's axioms we shall call *ordinary points* and *ordinary lines.* We must remember that they are undefinable except as the entities to which those axioms apply.

* " Line " means " straight line ".

We shall now introduce *special points** and a *special line*, distinct from ordinary points and lines, which again are undefinable except as those things to which any axioms we make about them apply. For the present, the terms *point* and *line* are to include ordinary and special points and lines, respectively.

We now assert :

($\alpha'$) *Any two ordinary points, or any ordinary and any special point, determine a single ordinary line containing them both.*

($\beta'$) *Every ordinary line contains one and only one special point.*

($\gamma'$) *Any two ordinary lines meet at a single point. If they are not parallel, the point is ordinary ; if they are parallel, the point is special.*

The *self-consistency* of ($\alpha'$), ($\beta'$), ($\gamma'$) is fairly clear. To see their *equivalence* to ($\alpha$), ($\beta$), ($\gamma$), we note firstly that if two ordinary lines meet in an ordinary point, they are not parallel, for if they were, by ($\gamma'$), they would meet in a special point and therefore not in an ordinary point. Similarly, if two ordinary lines meet in a special point they are parallel. Next, let $l$ be any ordinary line and $P$ any ordinary point not on $l$. By ($\beta'$) there is a unique special point on $l$, and by ($\alpha'$) there is a unique line joining this to $P$, and by what has just been said, this is parallel to $l$. But this result is precisely ($\gamma$). Thus we see that ($\alpha'$), ($\beta'$), ($\gamma'$) contain everything contained in ($\alpha$), ($\beta$), ($\gamma$). It is seen further that they contain no more. For example, there is no ordinary line joining a special point, not on $l$, to the special point on $l$, since such a line would contain two special points and so contradict ($\beta'$). Therefore ($\alpha'$), ($\beta'$), ($\gamma'$) merely give us another way of saying the same thing as ($\alpha$), ($\beta$), ($\gamma$).

Now, if ordinary lines $m$, $n$ are each parallel to an ordinary line $l$, they each meet $l$ in a special point. But there is only one special point on $l$. Therefore $m$, $n$ have the special point in common ; therefore they are parallel. Thus we recover the familiar euclidean theorem : Two lines which are parallel to the same line are parallel to one another. This is an example of the convenience of ($\alpha'$), ($\beta'$), ($\gamma'$). This we shall now consider in a more general way.

Desargues' theorem on perspective triangles, with various special cases in which certain lines are parallel, is a theorem of metrical euclidean geometry of the plane.† It can be regarded as a test of concurrence in the form, in euclidean nomenclature : A necessary and sufficient condition for three lines $a$, $b$, $c$, not all parallel, to be concurrent is that points $A_1$, $A_2$ on $a$ ; $B_1$, $B_2$ on $b$ ; $C_1$, $C_2$ on $c$ can be found such that $A_3$, $B_3$, $C_3$ are collinear, where $A_3$ is the meet of $B_1C_1$, $B_2C_2$, etc. But the theorem states also that if $a$, $b$, $c$ are all parallel then triads $A_1$, $B_1$, $C_1$ ; $A_2$, $B_2$, $C_2$ exist having this property, and conversely that if such triads exist then $a$, $b$, $c$ are concurrent *or* are all parallel. Passing now to the nomenclature of

*This is merely an *ad hoc* term, and should not be confused with its use, sometimes made, for the Absolute Points or Circular Points.

† It can be proved by means of the theorems of Ceva and Menelaus.

$(\alpha')$, $(\beta')$, $(\gamma')$, we have seen that if $a$, $b$, $c$ are all parallel, they have a special point in common. So all we have just said condenses to: A necessary and sufficient condition for ordinary lines $a$, $b$, $c$ to be concurrent is that ordinary points $A_1$, $A_2$ can be found on $a$, $B_1$, $B_2$ on $b$, $C_1$, $C_2$ on $c$, so that $A_3$, $B_3$, $C_3$ lie on an ordinary line, where $B_1C_1$, $B_2C_2$ meet in the ordinary point $A_3$, etc. By "concurrent", we now mean, meet in a point, ordinary *or* special. Similarly we can show that this meaning allows us to state any similar test of concurrence, without exceptional cases involving parallelism. This might have been foreseen without a detailed example, but we shall now find it useful to consider the dual example.

"*Special line*". The dual statement of Desargues' theorem can be regarded as a test of collinearity in the form (euclidean nomenclature): a necessary and sufficient condition for three points $A_3$, $B_3$, $C_3$ to be collinear is that we can find lines $l_1$, $l_2$ through $A_3$, $m_1$, $m_2$ through $B_3$, $n_1$, $n_2$ through $C_3$, such that $A_1A_2$, $B_1B_2$, $C_1C_2$ are concurrent, where $m_1$, $n_1$ meet in $A_1$; $m_2$, $n_2$ meet in $A_2$; etc. But Desargues' theorem states also that if $l_1$, $l_2$; $m_1$, $m_2$; $n_1$, $n_2$ are respectively parallel pairs of lines then $A_1A_2$, $B_1B_2$, $C_1C_2$, defined as before, are concurrent or parallel, and conversely that if the latter are concurrent or parallel, then $A_3$, $B_3$, $C_3$ exist and are collinear, or $B_3$, $C_3$ (say) exist and $l_1$, $l_2$ are parallel to $B_3C_3$, or $l_1$, $l_2$; $m_1$, $m_2$; $n_1$, $n_2$ are all parallel pairs. Passing now once again to the nomenclature of $(\alpha')$, $(\beta')$, $(\gamma')$, we see that in the last case $A_3$, $B_3$, $C_3$ exist as the *special* points on $l_1$, $m_1$, $n_1$, and with this description the rest of the theorem now shows that if $A_3$, $B_3$, $C_3$ are *any* three special points in the plane they satisfy formally exactly the same necessary and sufficient collinearity conditions as three *collinear* ordinary points. This result could be shown also for any other similar test of collinearity that might be proposed.

Therefore, we should get consistent results if we assert:

($\delta'$) *All the special points in the plane lie on a single special line. All points of the special line are special. It meets any ordinary line in the special point on the latter.*

In particular, we could now state Desargues' theorem completely in the form: If two triads of points are in perspective from a point, they are also in perspective from a line, and conversely. It is then true for any possible configuration of ordinary and special points and lines, these possible configurations corresponding exactly to all the particular cases considered in the euclidean formulation. Similarly, we should find that we can get such a general statement of any "projective" theorem in the geometry.

We should find, however, that we cannot attach "metrical" properties to the special points. On any ordinary line, we cannot ascribe a length to the segment between a given ordinary point and the special point. Nor can we ascribe a length to any segment of the special line. Also we cannot speak of the angle between the special line and an ordinary line, *i.e.* we cannot ascribe a direction to the

special line. (We can, however, work consistently with the assumption that the special point on an ordinary line divides every ordinary segment of it, *i.e.* one not containing the special point, externally in the ratio unity; but we shall not go into this.)

What we really discover in this development is the important fact that the metrical theorems are precisely those for which the special points and the special line *are* special. This will be fully appreciated by the student when he comes to study the reverse process of the metrical expression of the results of projective geometry.

*Points at infinity and the line at infinity.* What we have been calling *special points* and the *special line* are, of course, what are usually called respectively *points at infinity* and the *line at infinity*, and we now change over to this more customary nomenclature. The only thing in our approach which renders these terms at all natural is the fact that we cannot assign to any " point at infinity " a *distance* from any other point. We have not introduced these at the outset, since we wanted to avoid using the conception of a point at infinity as the " limiting position " of an ordinary point when its distance from some other fixed point tends to infinity. This is very apt to lead to misunderstandings, particularly in regard to the line at infinity, for it makes it difficult to dissociate the idea of " direction " from this line.

*Ranges and pencils.* The next step is to show how the points at infinity and the line at infinity may be included in the theory of ranges and pencils. If *cross-ratio* and *involution* are defined metrically, this can be done with the aid of the usual theory of projection from a point in the plane. In particular we may now define *the orthogonal involution on the line at infinity* (to be referred to briefly as the *orthogonal involution*). For parallel rays of all orthogonal involution pencils meet in the same point on the line at infinity, and so all such pencils define the same involution on the line at infinity. We see that a necessary and sufficient condition for two lines to be at right angles is that they should cut the line at infinity in a pair of mates of the orthogonal involution. This involution has no double points, since an orthogonal involution pencil has no double lines.

*Circles and the " Circular Points ".* A pair of parallel tangents to a given circle meet at a point at infinity, $\Omega$ (say). We may therefore regard the diameter joining the points of contact as the polar of $\Omega$. Since it is perpendicular to the tangents, it meets the line at infinity in the mate $\Omega'$ of $\Omega$ in the orthogonal involution. So $\Omega, \Omega'$ are conjugate points with regard to the circle. Therefore the mates of the orthogonal involution are conjugate pairs of points with regard to *all* circles in the plane.

Now, an involution has either no double points, or two distinct double points. *If* the orthogonal involution had double points, $I, J$ (say), each would then be self-conjugate with regard to every circle in the plane. Also, if a point is self-conjugate for a circle, it lies on the circle. So all the circles in the plane would pass through $I, J$. (Or we might reach the same conclusion in more detail by

analogy with the involution on the radical axis of a coaxal system of circles). Thus, if $I$, $J$ exist, they have the properties customarily ascribed to the " circular points ".

However, we have seen that the orthogonal involution has no double points, so $I$, $J$ do not exist. They can be introduced only by an extension of the geometry to include imaginary elements suggested by the generalisation of the algebraic representation. This can, of course, be carried through in a rigorous manner. But then the geometry is no longer the Real Geometry with which we started. If we wish to keep to this—and it is much better to complete its study as a self-contained system before considering other systems—we do not ever define the " circular points ". The properties sometimes related to these points, and which are a significant part of euclidean real geometry, can all be expressed in terms of the orthogonal involution itself. This is illustrated below by simple examples on conics.*

It must be emphasised that there is a fundamental distinction between the introduction of " points at infinity " and any proposed introduction of the " circular points ". The former give convenient expression to features already present in the geometry. They do, in fact, *exist*, in the only relevant sense of the word, as satisfying the assumptions which we have made about them and shown to be contained in Euclid's axioms. The " circular points " on the contrary, do not exist, since we can show that the orthogonal involution has no double points.

*Conics.* Conics may be supposed to have been introduced by any of the standard definitions. If the definition by sections of a cone is used, the geometry must, of course, have been extended to three dimensions, and in particular, for present purposes, the *plane at infinity* must have been introduced by showing that the lines at infinity in all planes satisfy formally the standard conditions of coplanarity. The relation of the *vanishing line* to the *line at infinity*, in projecting from one plane to another, must then have been indicated. While this definition of conics is in some ways the most pleasing, it seems on the whole more desirable first to continue the geometry of the plane, by using some definition which does not require us to go outside the plane.

The distinction between the ellipse, parabola, and hyperbola, by means of their relation to the line at infinity can then be established. It can next be shown that the usual generation of a conic by the intersection of corresponding rays of two projective pencils also holds good if one or both pencils have their vertices on the line at infinity. Other extensions of the projective properties to include points at infinity may be considered.

It is then easy to prove :

*Theorem. A necessary and sufficient condition for a conic to be a*

* Views of a similar tenor, though not explicitly in regard to the " circular points ", were expressed by Professor Neville in a recent address on " *Negative Squares in Real Geometry* " to the Mathematical Association of N. Ireland. See also A. Robson, *Gazette* 20 (1936), 51.

*circle is that all pairs of mates of the orthogonal involution are conjugate in regard to it.* The condition is more usually stated as, " the conic should pass through the circular points ". But in the latter form, it is not a theorem of the present geometry.*

*Foci of conics.* We adopt the definition : *A focus $S$ of a conic is a point, such that every pair of conjugate lines through $S$ is orthogonal.* This can be proved equivalent to the " focus and directrix " definition. We first prove, by a simple consideration of the diameter through $S$ and its conjugate chord :

*Theorem* 1. *If a focus exists, it must lie in a principal axis.*

We then prove :

*Lemma.* $\Gamma$ is a given conic ; $P$, $Q$ are fixed points, not conjugate w.r.t. $\Gamma$ and $PQ$ does not touch $\Gamma$ ; $PR$ is a variable line through $P$, and $QR$ the conjugate line through $Q$, meeting $PR$ in $R$. *The locus of* R *is a (non-degenerate) conic $\Sigma$ through* P, Q. *The points of intersection of $\Sigma$, $\Gamma$ are the ends of the chords of contact of* P, Q *w.r.t. $\Gamma$, if these points exist.*

*Cor.* (*a*). If $PQ$ touches $\Gamma$ in $T$ (say), the locus of $R$ is the line joining the point of contact of the other two tangents from $P$ and $Q$, together with the line $PQ$ itself.

*Cor.* (*b*). If $P$, $Q$ are conjugate w.r.t. $\Gamma$, the locus of $R$ consists of the polar lines of $P$ and $Q$.

Now, let $P$, $Q$ be a pair of mates of the orthogonal involution. The lemma then becomes :

*Theorem* 2. *Pairs of conjugate lines w.r.t. an ellipse or hyperbola $\Gamma$, which are parallel to two fixed perpendicular directions, intersect on a rectangular hyperbola $\Sigma$, concentric with $\Gamma$, which meets $\Gamma$ at the ends (if they exist) of the diameters conjugate to these two directions.*

The fixed directions are not parallel to the axes of $\Gamma$, for that is the case of Cor. (*b*).

Thence it follows :

*Theorem* 3. *An ellipse or hyperbola has two and only two foci, and these lie symmetrically on one of its axes.*

For since the axes are at right angles, one axis meets the rectangular hyperbola $\Sigma$ in two points $S$, $S'$, equidistant from the centre, and the other does not meet $\Sigma$. Through $S$, there is a pair of conjugate lines parallel to the fixed directions, and also a pair parallel to the axes. So there are two pairs of orthogonal conjugate lines. Therefore, by a known theorem, all pairs of conjugate lines through $S$ are orthogonal. Similarly for $S'$. Therefore $S$, $S'$ are foci.

It now follows that all rectangular hyperbolas† like $\Sigma$ go through $S$ and $S'$ and that there are no other foci.

* An instance of a different and more blatant misuse of the circular points was shown to me by my colleague, Dr. R. Cooper. A student asserted " The circles $|z|=a$, $|z|=b$, in the complex plane, intersect in the circular points "!

† These include, and give a geometrical interpretation of, the rectangular hyperbolas found in one of the standard analytic methods of getting the foci, *e.g.* Sommerville, *Analytical Conics* (1933), 139, Equations (3), (4).

It is then not difficult to show that, if $\Gamma$ is an ellipse, the foci lie on its major axis, and if $\Gamma$ is a hyperbola, its foci lie on the axis which meets $\Gamma$.

By a similar treatment of Cor. (*a*), and Cor. (*b*), we get, respectively :

*Theorem* 4. *A parabola has one and only one focus, and this lies on its axis.*

*Theorem* 5. *A circle has one and only one focus, and this coincides with its centre.*

This treatment or something analogous to it should, I suggest, replace in this geometry, the one which is customarily based upon the statement " the foci of a conic are the remaining points of intersection of the tangents through the circular points ". These tangents do not exist in the Real Geometry, nor do the imaginary foci, and to derive the real foci by a method which goes outside the real geometry is unsatisfactory. The present treatment might seem lengthy, but it should be noticed that it deals in a natural manner with the various particular cases *en route*.

[In the discussion Mr. T. L. Wren said that the treatment contained in Theorems 1-5 is the one he has used for a number of years and pointed out that it is given in Filon, *Projective Geometry* (4th edition, 1935), 145-7. Nevertheless I have ventured to retain these theorems here, since their explicit statement helps to make clear the point of my argument.]

**Professor H. S. Ruse** (University College, Southampton) :

### Differential Geometry

My contribution to this discussion reminds me of the ill-prepared Divinity student, who, asked in an examination to " Discuss the textual criticism of the Minor Prophets ", began, " Far be it from me to criticise these great and holy men ; be it mine rather to give the list of the Kings of Judah and Israel". For what I have to say has about the same degree of relevance to the topics discussed by the last two speakers as the student's answer had to the question asked, and my remarks will, moreover, be more in the nature of a sermon than of a mathematical lecture. I hope, however, that they will serve as an illustration of the general subject under discussion, even though they may not bear very directly upon what has already been said.

I should say at the outset that my object is to seek rather than to try to impart information, because, having had no personal experience of teaching in schools, I am unable to give advice to those actually engaged in the difficult task of instructing beginners. I am hoping to learn from the discussion whether the present-day examination system allows the teacher to wander occasionally outside the confines of examination syllabuses and to give his pupils

some vision of wider cultural horizons, or whether, as observation of university freshmen seems to suggest, secondary education has come to mean no more than a weary plodding along the narrow way that leads to Matric or Higher School. " Only do enough examples, my boy,* and you'll get through " appears to be the key-note of the instruction given to many young students of mathematics, and they may never realise that their subject has any essential connection with the world around them. Even their applied mathematics is concerned with stereotyped problems about weightless elephants, perfectly elastic billiard balls and the like ; and, although they may have learnt through the unsafe medium of popular books on science that Relativity and Quantum Mechanics have revolutionised the scientific and philosophical thought of our time, they apparently find little or no reflection of this fact in their school work. It is probably no exaggeration to say that, to all except a few of special ability who appreciate its intrinsic beauty, mathematics remains a matter of learning rules for the solution of puzzles of a conventional type, unrelated either to the social problems confronting our civilisation or to the more purely intellectual questions of modern science.

In saying that I do not wish to suggest that a drilling in problem-solving can, or should, be avoided, for the technique of the subject can hardly be acquired in any other way. No branch of learning can be studied without a certain amount of initial drudgery. But I do wish to suggest that some endeavour be made to get away from the spirit of Todhunter and to acquire—dare I say it?—something of the spirit of Hogben, whose *Mathematics for the Million*, though perhaps hardly a suitable textbook for anyone wanting a systematic knowledge of the subject, nevertheless does present mathematics as a living part of human knowledge and successfully de-bunks those who would make a mystery of it. In parenthesis, let me express the hope that some grateful mathematician will do for Hogben what he has done for us, and write a *Biology for the Bumpkin* ; I am sure that such a book is needed.

Differential geometry provides a useful example of how recent advances in mathematics and physics can be made to influence teaching both in school and in universities. Forming as it does the mathematical basis of the General Theory of Relativity, it has developed rapidly since the inception of the latter theory in 1915, and has now become one of the largest and most important branches of pure mathematics. Yet the ordinary textbook gives no indication of this fact. Elementary differential geometry (by which I mean the theory of tangents, normals, arc-lengths, curvature, and so on) is still treated as a mere illustration of the calculus, and is carefully separated from its other parent, analytical geometry. No attempt is made to fire the student's imagination by anticipatory suggestions about the higher developments of the subject, and little indication

* Or its feminine equivalent, whatever that may be. It is a curiosity of our language that while " my boy " sounds paternal, " my girl " sounds definitely offensive.

is given of its connections with other branches of mathematics or with other sciences.

Higher differential geometry may be described for present purposes as the intrinsic geometry of curved spaces of any number of dimensions. The invention of a specially devised calculus (tensor analysis) has made it possible to express its results in a form valid for any system of coordinates, and in it the idea of a vector is fundamental.

How much of that brief statement, I wonder, would be intelligible to a boy or girl specialising in mathematics at school? Probably very little. What, it might be asked, is *intrinsic* geometry? What is meant by a curved space? And how can there be more than three dimensions? Are there any systems of coordinates other than cartesians or polars? And, finally, what have vectors, which surely only enter into mechanics and to some extent into pure geometry, to do with the sort of geometry one meets with in books on the Calculus?

It is possible, I think, to enlarge upon the ordinary elementary analytical and differential geometry in such a way that these questions could easily be answered, and so that the inquiring student could read the more authoritative of the semi-popular books on Relativity, like those of Einstein himself, with real profit and understanding. I wish therefore to devote the brief time at my disposal to suggesting how this might be done. In doing so I have in mind the boys and girls at school who intend to proceed to a university to take an honours degree in mathematics, and also first-year students at the universities themselves. As a warning, I should perhaps say that I shall be making no suggestions of a startling nature, and that I am not particularly concerned with actual methods of teaching. My sole object is in fact to describe the sort of general ideas that might be presented to the student in anticipation of his later work, without saying much about the precise manner in which the mathematical details could be worked out. If I preach to the converted, so much the better ; and if I seem superficial, I apologise and make the usual excuse that there is insufficient time for profundity.

Quite early in his career as a mathematician, the student is introduced to rectangular cartesian coordinates and is taught the usual analytical geometry of the straight line, the circle, and elementary conics. He is also told about polar coordinates and learns something of their use. Now as soon as he is familiar with these particular systems, his mind can be broadened by a general talk about coordinates. First of all, it can be pointed out that the coordinates of a point in a plane are, by definition, *any* set of numbers, usually two, that serve to fix its position in relation to some frame of reference (in cartesians the axes and in polars the pole and initial line), or, more significantly, in relation to some mesh-system. Emphasis can be laid on the fact that the mesh-system appropriate to polar coordinates is not the ordinary cartesian squared paper, but consists of a set of lines radiating from the pole, together with a set of *curves*,

namely concentric circles having the pole as centre. If possible the student should be provided with polar graph-paper and encouraged to draw for himself some of the well-known and very beautiful polar curves like the spirals and epicyclics. In this way his mind will be prepared for the idea of curvilinear coordinates in general—coordinates in which the mesh-system consists of any conveniently chosen sets of curves. He can in fact be encouraged to invent coordinate systems for himself, and to find the shape, say, of a curve given by a linear equation in bipolar coordinates. As an illustration, he could be shown that a system of curvilinear coordinates for the upper half of the $(x, y)$-plane is obtainable by taking the systems of coaxal circles $x^2+y^2+2\lambda x=0$, $x^2+y^2+2\mu y=0$ and specifying a point by the particular pair of circles passing through it, so that $(\lambda, \mu)$ become the coordinates of the point; and again he could be encouraged to find the shape of curves given by such equations as $a\lambda+b\mu+c=0$, and so on.

He can next be introduced to the idea, important in later work, of regarding a transformation from one set of coordinates to another, *e.g.* from one set of cartesians to another or from cartesians to polars, not as a mere renaming of the same point in a plane, but as a way of mapping one plane upon another or upon itself. Thus, for example, there is no inevitable necessity for interpreting $(r, \theta)$ in the equations $x=r\cos\theta$, $y=r\sin\theta$ as polar coordinates in the $(x, y)$-plane: they may equally well be regarded as cartesian coordinates in another plane, and the equations as providing a correspondence, though not (1, 1), between the two planes. If we write $(X, Y)$ for $(r, \theta)$, we have $x=X\cos Y$, $y=X\sin Y$, and a straight line such as $x+y=1$ corresponds to the curve

$$X(\cos Y+\sin Y)=1,$$

*i.e.*

$$X=\frac{1}{\sqrt{2}}\operatorname{cosec}(Y+\frac{\pi}{4}),$$

and so on; while if we restrict $Y$ to lie between $-\pi$ and $+\pi$, we actually obtain a (1, 1) correspondence between a strip of the $(X, Y)$-plane and the whole of the $(x, y)$-plane. As an illustration of mapping this particular example is of no great interest, but the student could be given, say, some simple conformal transformation, and asked to find what happens to straight lines, circles and other curves when they are subjected to it. He could also be told in a general way about conformal representation and the mathematical ideas underlying map-making, and, if he were sufficiently advanced in knowledge, about the way in which a linear transformation of homogenous coordinates can be interpreted either as a simple change of coordinates or as a kind of mapping, called a collineation, that arises in non-metrical projective geometry. Further, those acquainted with tangential coordinates and the consequent idea of attaching coordinates to geometrical entities other than points could be recommended to consider the effect of regarding $g, f, c$ in the

equation $x^2+y^2+2gx+2fy+c=0$ as the coordinates of the circle, while those knowing a little three-dimensional analytical geometry could be shown how the simple expedient of interpreting $(g, f, c)$ as the coordinates of a point in three dimension yields a variety of interesting results.* Such examples as these would give to the student some understanding of the freedom enjoyed by geometers in the interpretation of algebraical formulae and an altogether wider conception than he could otherwise have of the range and power of analytical methods.

Allied to these ideas is that of the dimensions of a space. What, the student could be asked, does he understand by the statement that a plane is two-dimensional and that " space " is three-dimensional? He can quickly be made to realise that it is a mere question of terminology. A plane is *called* two-dimensional because at least two numbers (coordinates) are required to specify the position of a point upon it in relation to a given mesh-system, and space is *called* three-dimensional because at least three such numbers are required. Shall we then call the surface of a sphere two-dimensional or three-dimensional? It is a commonplace that only two numbers, *e.g.* latitude and longitude, are required to determine the position of a point upon it, and it is therefore natural to call this surface two-dimensional in spite of our normal habit of regarding a sphere as a three-dimensional object. At this point the student can be introduced to the concept of the intrinsic geometry of a surface—that is, the geometry of the surface itself as apart from its relation to the surrounding three-dimensional space—and can easily be persuaded to think of the surface of a sphere or of any other solid as a *two*-dimensional region, or space, having a geometry of its own. Reference can be made to a well-known particular case of the intrinsic geometry of a curved two-dimensional space, namely spherical trigonometry, in which the sum of the three angles of a triangle is not equal to two right angles, in which there is no parallelism in the ordinary sense, and in which a number of the other familiar concepts of plane euclidean geometry do not hold. From this point of view the ordinary plane geometry of the textbooks is just the special case of two-dimensional intrinsic geometry which arises when the space happens to be flat. In this connection some account of theories of parallelism and non-euclidean geometry might also be given.

With general remarks of this kind could be associated some actual mathematics if the student were sufficiently mature. It could be pointed out that the familiar formula

$$ds^2=dx^2+dy^2 \qquad (1)$$

is peculiar in that the right-hand side consists of a sum of squares of differentials, and that it ceases to have this simple form for systems of coordinates other than rectangular cartesians. Thus in polar coordinates it becomes

$$ds^2=dr^2+r^2d\theta^2,$$

* See D. Pedoe, *Math. Gazette*, vol. 21 (1937), p. 210.

and in a general system of curvilinear coordinates $(u, v)$ it has the form

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad (2)$$

with $E$, $F$, $G$ functions of $u$, $v$. Now the quadratic form (1), or the equivalent but more elaborate expression (2), contains within itself all the intrinsic geometry of the euclidean plane,* and all the standard formulae for tangents, normals, curvature, etc., can be expressed in terms of $u$, $v$, $s$, $E$, $F$, and $G$. But what, it could be asked, may be said about a quadratic form like (2) which is not reducible by a transformation of coordinates to the simple form (1)? The answer is, of course, that it may be interpreted as defining the intrinsic geometry of a two-dimensional space which is *not* plane ; that is, in more elementary language, of a curved surface.

The student can now be given some account of the differential geometry of surfaces and of the fundamental part played by the form (2) in that theory. He can be told how geodesics and principal curvatures are defined, and how some of the formulae are direct generalisations of those of the plane. He can be made aware of the existence of tensor analysis, which makes it possible to express the results in a form valid for any coordinate-system, and of the summation-convention, which renders the restriction to two dimensions unnecessary because the formulae remain the same whatever the number of variables. In this way he may obtain a real understanding of what is meant by a curved space of $n$ dimensions and by the " four-dimensional continuum " of Relativity. He will see that these ideas are reached by a natural extension of the analysis used in the ordinary differential geometry of the plane and of curved surfaces, and will realise that, although the visualisation of a curved space is difficult when its number of dimensions is three and impossible when that number is greater than three, yet there is nothing in the least mysterious about the concept of multidimensionality. It may indeed be hoped that the fourth dimension of Relativity would thus be divested of that spurious air of romance given to it by some novelists and writers of popular science, and be seen in its true light as a technicality introduced into physics purely for reasons of mathematical convenience. Meaningless questions such as " Is the universe *really* four-dimensional? " would perhaps no longer be asked, for it might be realised that the number of dimensions of space-time is not a physical actuality awaiting discovery, but an abstract idea belonging to that particular description of physical phenomena which is called the Theory of Relativity.

Mention has already been made of the fundamental part played by vectors in higher differential geometry, and by way of example we may note that in advanced work one does not ordinarily speak of the direction-cosines of the tangent to a curve, but of the unit tangent-vector at a given point. This, roughly speaking, is because the direction-cosines can only be thought of as a discrete set of

* I am assuming that euclidean parallelism is taken as fundamental.

numbers depending upon the particular coordinate-system used, whereas a vector can be visualised as a single entity independent of that system. Thus in plane differential geometry, treated from the advanced point of view, the familiar statement that

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi,$$

where $\psi$ is the angle between the axis of $x$ and the tangent at the point $(x, y)$ to a curve of arc $s$, would be replaced by the statement that $(dx/ds, dy/ds)$ are the components of the unit tangent-vector at $(x, y)$. This is a rather trivial example, but it does suggest that some instruction in vector analysis might well form a part of the early education of every mathematical specialist. Most boys and girls become familiar with the idea of vectors in mechanics, and also, perhaps, in their reading of books like Durell's *Modern Geometry*, but I imagine that comparatively few are taught vector analysis as such, and that fewer still have any conception of the important place occupied by vectors in advanced branches of mathematics like the tensor calculus and the theory of matrices. That the subject is still not widely taught in schools or universities perhaps accounts for the persistence in British mathematical literature of the obsolete bracket-notation for scalar and vector products, and for the evident fact that many prominent mathematicians regard it merely as the source of a convenient shorthand notation for physicists and not at all as the powerful calculus that it undoubtedly is. I therefore feel quite strongly that it should be introduced to students at quite an early stage, and that its methods should be employed as often as possible in geometry and mechanics. This would involve no very extensive addition to the ordinary curriculum : the student would already be familiar with the parallelogram law of addition, and it would take very little time to introduce him to the Clarendon " black-type " notation and to teach him the theory of scalar and vector products. The more advanced calculus associated with the operator $\nabla$ could be postponed for future study at the university. Even so slight a knowledge of vector analysis would, in my opinion, be extremely useful to him in his later work, and would give him a fuller understanding of the methods and point of view of higher branches of mathematics. Incidentally, it may be noticed that, inasmuch as

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

when $\mathbf{A}$ and $\mathbf{B}$ are non-zero vectors and the vector product is taken, the subject provides a simple example of an algebra in which the commutative law of multiplication does not always hold. So here again, the student's ideas could be widened by making him realise that there is nothing sacrosanct about the fundamental laws of ordinary algebra, and that perfectly logical systems can be developed in which some of those laws do not hold at all or are replaced by other laws.

I have purposely not referred to the desirability of giving young students preliminary notions about the differential geometry of twisted curves in three dimensions, mainly because I could hardly do so effectively without entering into mathematical details. I have wished to emphasise the general rather than the particular, and to suggest that students might be given some idea of the shape, if not of the inner workings, of the interesting things to come. No doubt many of them are already given such glimpses of their mathematical future, and it is probable that my remarks will have seemed extremely commonplace to those who know more than I of school mathematics. But the fact remains that many, if not most, of the new arrivals at the universities appear to be passive receptacles of selected information, having little idea of what lies before them and suffering from the mental ossification that comes from being fed on the dry bones of their subject without any of the meat. Some, alas, remain mentally inert to the very end. It is no secret that teaching in the universities, given as it is by people with no special training and often with little aptitude, is frequently inferior to that in the schools. The secondary kettle is in no way blacker than the university pot, and those who blame the schools for everything that is wrong in the university classroom are in danger of spattering themselves with the mud they fling at others. It is a depressing fact that it is still possible to obtain a degree at certain of our institutions of higher learning without knowing anything of contour integration or Cauchy's theorem, and without having heard of matrices or tensors: one might indeed imagine from a perusal of their syllabuses that mathematical research had ceased about the middle of last century. So although my remarks, for what they are worth, have been addressed mainly to teachers in schools, I am fully conscious of the need of a broader concept of mathematical education in some of the universities themselves. I am not of course advocating the replacement of a sound foundation of mathematical knowledge by a woolly waste of smatterings, nor am I suggesting that manipulative ability is to be despised; but I do think that it should be possible, even in the face of parental and gubernatorial demands for "results", to brighten the schoolboy's intellectual horizons by telling him something about the higher developments and applications of his work, and to give him something of what may be called the research-outlook. The mote in the examiner's eye will not be removed until the teacher (who is, after all, sometimes the same person) has plucked the beam from his own eye. I therefore conclude, as I began, by expressing the hope that the wider cultural and social aspects of mathematics will not be lost sight of amid the struggle for scholarships and the stultifying scramble for certificates which pass in many places for education, and that students will be encouraged to look sometimes at the mathematical wood as well as at the individual trees.

# HYPERSOLID CONCEPTS, AND THE COMPLETENESS OF THINGS AND PHENOMENA.*

By Cyril H. H. Franklin.

The fundamental idea expressed in this paper is that objects and phenomena are only seen in part, vision being limited to a plane in time about $\frac{1}{20}$ second thick, and showing us things as they are for the instant only: but if we wish to picture the Completeness of Things, or Phenomena, we have to include, certainly, time as well as space; we have to retain the image of the movement of the rolling ball, the running machine, the growing plant, the dancing girl, through either the cycle or phase of movement or the life history, as appropriate.

To take a concrete example, the completeness of an apple is more than the spheroid seen on the plate; it has been growing to maturity for, say, four months, and will continue to exist until its break-up in one of the familiar ways. Obviously it begins as the blossom which falls away and reveals a tiny fruit which duly expands to maturity, and so on. It may be photographed daily and a cinema film produced showing the growth in a few seconds; or the completeness of the apple in the space-time continuum may be pictured as a cigar-shaped hypersolid figure whose section on any given instant hyperplane will be the apple as seen normally by the eye.

But an apple which merely existed in space-time would be rather inadequate as food, showing that a greater degree of completeness is obtainable and indicating the need for a space-time-mass continuum; we may continue our expansion and the apple becomes a figure very inadequately described as a five-dimensional cigar shape, which, sectioned again on the time hyperplane, becomes, approximately an oblate hyperspheroid.

An objection may be raised that while time and space obviously lie both before and behind an observer, mass does not. This is largely a matter of the orientation of consciousness, and careful consideration will show that we know mass relative to ourselves in two ways, first by inertia, and secondly by momentum, which are opposite in effect to the observer.

A preliminary statement as to the completeness of things may now be made, that they are dimensionally at least

$$S^3TM.$$

When we study phenomena this stage of completeness is again realised to be inadequate, as no recognition of essential qualities such as are obvious in many circumstances is made. Typically, some of these are

* Summary of a paper and demonstration to Section A of the British Association, Nottingham, 7th September, 1937.

| | |
|---|---|
| Electrification, $E$ ; | Magnetization, $m$ ; |
| Heat, $H$ ; | Stress, $f$ ; and so on.* |

Once again, these may be noted as bidirectional relative to the observer. Expressing these qualities as part of the completeness, the picturing of things becomes

$$S^3TMHEmf,$$

which requires a rather appalling technique for a full expression of the dimensionality involved.

It is sufficient for immediate advance to give consideration to four-dimensional geometry, with a little introduction to five- and six-dimensional space, and the means of representing them pictorially, for only when geometry is expressible in a manner comparable with ordinary experience is it usable by the majority of us ; a picture drawn with perspective inverted would be incomprehensible to most of us and confusing to the skilled geometrician.

Ultimately, everything to be freely comprehensible must be expressed in terms of our ordinary life experience ; we are the centres of our worlds and we may spread out as many Cartesian axes as we find necessary to the framework of our Universe. Normally, we are accustomed only to the three Cartesian axes of ordinary space, mutually at right angles, and objects falling into their natural perspective around them as vision shows. But the completeness of things, and corresponding transcendental thinking, requires more ; spacial concepts and their appropriate perspective have to be developed to suit. However, we are accustomed to record three-dimensional experiences in two dimensions, as every illustrated paper, picture or photograph shows ; and in the main we are able to interpret these two-dimensional distortions of things back into approximately the three-dimensional objects recorded with reasonable facility. Hence it should be possible to develop a suitable technique for displaying experience in four or more dimensions in 2-space or 3-space, and to re-interpret our picture into the reality.

Looking into basic principles :

> In space of ordinary instantaneous experience, three lines only are possible mutually at right angles, the three Cartesian axes.
>
> In picturing the completeness of objects or phenomena any number of Cartesian axes mutually at right angles may be required.
>
> In representing 3-space in a drawing or photograph (which is a 2-space projection) the three axes may have any angles mutually, but their sum will be 180°.
>
> Areas will be diminished proportionally, according to laws dependent on the type of projection used.

* No attempt is made to identify these with any of the units in ordinary dimensional theory ; they indicate directions loosely.

Lengths may be maintained if parallel to the axes, or diminished according to the type of projection.

Angles may be decreased or increased, dependent on their relation to the axes and on the type of projection.

Parallel lines may remain parallel, or may converge proportionately to distances from the observer according to the type of projection.

The following statements can now be regarded as Axioms for representing $N$-space in $(N-n)$-space :

1. Any number of Cartesian axes may be drawn through a given point, in space of any dimensionality above unity.
2. Any angles may be arranged between these axes.
3. Any rectilinear figure may be drawn with consistent distortion of lengths, areas, volumes and hypervolumes, values being dependent on the scheme of projection and consequent angles used.

As a basis for considering the representation of hypersolid space figuring, some methods of representing 3-space in 2-space may be recalled :

Perspective, which is ordinary visual experience.

Orthogonal projection, usual for machine drawing.

Oblique projection, often found in the untrained artist's sketch.

Isometric projection, often the plumber's sketch for pipe work.

Isogonal projection, all angles representing 90° being changed to 60° or 120°.

Heterogonal projection, the random angle sketch.

Stereographic projection, used in maps, crystallography, etc.

Schlegel projection, connecting points (like an electrical diagram) regardless of parallelism and angular values.

Pictures of hypersolids can be either plane or solid. In the former case it should be possible to represent in perspective, by placing an appropriate group of vanishing points and developing towards them. A solid perspective picture of a hypersolid can only be viewed correctly in one direction ; if turned through 180° the perspective will be inverted. This would seem to indicate that isogonal isometric projection would be the best available projection method. However, oblique and heterogonal projections are also useful and intelligible.

The best method of approach to constructing hypersolid figuring appears to be by choosing a suitable Cartesian axial group for the form to be developed and the dimensionality required, and either drawing them (in 2-space) or modelling them (in 3-space) ; then using this group as the frame on which to erect the form to be studied, using coordinates to construct it if practicable.

Starting with the simplest hypersolid concepts, we may consider a cube for a period of time, which is $S^3T$. The actual hyper-

cube may be drawn in a distorted form, with its distortions comparable with oblique projection, by moving a cube along a diagonal straight line. The figure so described obviously has a cube at each end, and each of the six squares as it is translated traces a rhombic hexahedron which represents one cube face of the 4-space hypercube, so indicating the eight cube faces required. This method can be considerably developed, and the type of hypersolids which are called Parallelotopes by Sommerville can be described by its means, and if desired could be photographically recorded with a suitable technique.

Random translation of this type may produce very distorted faces, and may be compared with viewing a cube or other orthogonal solid from a small angle to the normal to a face. Obviously there are optimum conditions for representation. If minimum distortion is aimed at, and all parts of the figure studied are to have consistent proportions, the axes should, if possible, be placed either in drawing or in model with isogonal relations, areas then having consistent values provided they lie in planes parallel to pairs of axes and similarly volumes, when limited by planes parallel to pairs of the same three axes.

Isogonal projection is possible in 2-space for any number of dimensions, because the circle can be divided into any number of equal parts. Isogonal projection is possible in 3-space only when axes can be arranged regularly, that is, can have solid angles of the same shape completely around each of them, and these angles fitting completely over the surface of a sphere. It follows that the only dimensions of space that can be displayed in isogonal models are those corresponding to such of the regular solids as have their vertices symmetrically paired across their centres, these paired vertices placing the axes.

Other dimensions can be dealt with by axes placed preferably with the aid of the vertices of the Catalan-Poinsot semi-regular solids, as embodied in the work of Paul Donchian (U.S.A.), whose modelling includes the 3-space envelopes of many of the hypercubes up to 24-dimensional space, using groups of axes placed by this means. This method naturally tends to be limited to such dimensions as can be met by " vertices/2 " of Catalan solids. However, as indicated in Axiom 1, any sheaf of lines will serve as Cartesian axes, but near approach to spherical symmetry is generally desirable.

Where no regular or semi-regular solid provides suitable pairs of vertices there is another arrangement which is useful and which can be adopted to a type of oblique projection available for any dimensionality, particularly useful for 5-dimensional space. This consists of axes placed as the ribs of an umbrella at any chosen angle to the polar axes. This method introduces as the envelopes of hypercubes modelled in 3-space, an infinite series of rhombohedra with a second order of semi-regularity of a polar and equatorial type ; the number of faces is $N(N-1)$, where $N$ is the dimension of space whose cube is enveloped.

Large drawings of these polar rhombohedra for 5-space and 8-space were displayed at the British Association meeting and the characteristics of this apparently new series will be published in detail later; the earlier members of the series are obviously the rhombic hexahedron, rhombic dodecahedron and rhombic icosahedron; models of these were shown.

Models were displayed representing the hypercube in 4-space and 5-space, "hypertetrahedron", "hyperoctahedron" and other figures; Cartesian axial systems, and distortions of 3-space regular solids in building isogonal 4-space models, etc.; also drawings of $(A+B)^4$ complete and dissected, and photographs of Donchian models, etc., were exhibited. Stereoscopic drawings of the 4-space hypercube and the octahedron equivalent (having sixteen tetrahedral faces) were shown, and also a model displaying the relationship which exists between the regular solids of 3-space.

These models and drawings displayed facts such as the arrangement of the five tetrahedral faces of the 4-space "hypertetrahedron" or pentahedroid, and the hypersolid angle meeting at each vertex, and also typical distortions to be expected in isogonal and oblique hypersolid projections.

Much further work requires doing on the simpler figuring of $N$-space; but, adequate models and drawings being available to develop the subject, it should become easy to picture and think directly in the space-time continuum, and so proceed towards consciousness of the Completeness of Things instead of being almost confined to the instantaneous appearance which appears to be our normal experience.

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